Duality Involutions, Representations, and Geometry

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Abstract

In this thesis we give an exposition of the theory of duality involutions, and within this context we present the results of two different research projects.

Loosely speaking, a duality involution on a category C is a self-adjoint contravariant endofunctor of C. A prototypical example of such is the usual notion of duality for finite dimensional vector spaces. We also consider duality involutions for bicategories, as defined by Shulman.

The first project concerns classification problems in symplectic linear algebra. In this part, we discuss results regarding the symplectic group and its Lie algebra, as well as work on systems of subspaces in symplectic vector spaces. In the language of duality involutions, symplectic structures are encoded as fixed point structures.

The second project is about the Morita bicategory of finite-dimensional $\mathbf{k}$-algebras and bimodules, and the representation pseudofunctor which sends an algebra to its category of representations. We show that this representation pseudofunctor is equivariant in a natural manner with respect to duality involutions which we define on its source and target.

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To the reader

I hope you find your way! For confusing or lacking exposition, I apologize. This thesis is a patchwork of various kinds of work, each having a different focus and flavor. In particular, substantial portions of pre-existing text have been inserted, rearranged, and welded together. These portions have been complemented by additional material in the form of expository background, further examples, and general results which serve to tie parts together. Although I should not call it a work in progress, this text is a static representation of a dynamic process of understanding. Despite this (inevitable) incompleteness, I hope that the general themes and ideas, and their *direction*, are visible.

Original work

The main original research contributions presented in this thesis are the results of Chapters 7 and 8, as well as the results in Part 3. This material is taken, in a copy-paste manner, from the two papers [**HLW19**] and [**LV19**]; these paper are original work by myself and my co-authors, Christian Herrmann and Alan Weinstein, and Alessandro Valentino, respectively. We share the credit (and responsibility) equally. The exposition in Chapter 6 is my own adaptation of ideas, results, and techniques used in [**HLW19**]. The classification results themselves are mainly known, although perhaps not quite in this form. Some of Part 1 was worked out in collaboration with A. Valentino, and some is taken or inspired from the references [**QSS79**], [**Knu91**], [**Shi12**], [**Jac12**], [**FH16**], though I am unaware of a reference for a general theory of duality involutions.

Self-plagiarism

As mentioned above, parts of this thesis are copies of parts of the papers [**HLW19**] and [**LV19**]. Specifically, Chapters 7 and 8 are copied (with minor changes) from [**HLW19**], while Chapters 9, 10, and 11 are copied (with minor changes) from [**LV19**].

Introduction

The loose theme of this thesis is how "duality" relates to notions of geometry on the one hand, and of representation theory on the other.

The concept of duality that we use is formulated in the language of category theory, and we work with several related formalizations. The core basic notion is as follows: given a category C, a "duality involution" is encoded as the data of an adjoint equivalence

$$\mathsf{C} \underset{\delta^{op}}{\overset{\delta}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{C}^{op}, \tag{1}$$

where if $\eta : 1_{\mathsf{C}} \Rightarrow \delta^{op}\delta$ is the unit of the adjunction, then $\eta^{op} : \delta\delta^{op} \Rightarrow 1_{\mathsf{C}}$ is the co-unit.[1] We write this data as $(\mathsf{C}, \delta, \eta)$. The prototypical example for this definition is the case where $\mathsf{C} = \mathsf{vect}_{\mathbf{k}}$, the category of finite-dimensional vector spaces over a field $\mathbf{k}$, and where $\delta : \mathsf{C} \to \mathsf{C}^{op}$ is the usual duality for vector spaces: on objects,

$$\delta V = V^* = \mathrm{Hom}(V, \mathbf{k}), \tag{2}$$

and given a morphism $f : V \to W$, its dual is the usual adjoint $\delta f = f^* : W^* \to V^*$, i.e.

$$(f^*\xi)(v) = \xi(fv) \qquad \text{for } \xi \in W^*,\ v \in V. \tag{3}$$

This functor forms an adjoint equivalence of the kind (1), the unit of which has components

$$\eta_V : V \longrightarrow V^{**},\ x \longmapsto (\xi \mapsto \xi(x)). \tag{4}$$

The relation between duality and "geometry" appears in this text via the notion of a symmetric or skew-symmetric bilinear form on a vector space V. Such a form is a bilinear map

$$B : V \times V \longrightarrow \mathbf{k} \tag{5}$$

for which

$$B(x, y) = B(y, x) \quad \forall x, y \in V \tag{6}$$

(in the symmetric case), or

$$B(x, y) = -B(y, x) \quad \forall x, y \in V \tag{7}$$

(in the skew-symmetric case). To any bilinear map (5) we have associated maps

$$b : V \longrightarrow V^*, \quad x \longmapsto (y \mapsto B(x, y)) \tag{8}$$

and

$$b^t : V \xrightarrow{\eta_V} V^{**} \xrightarrow{b^*} V^*, \quad x \longmapsto (y \mapsto B(y, x)). \tag{9}$$

We will take the point of view that the maps (8) and (9) are fundamental, rather than (5). This allows for the following generalization from vector spaces to any category with duality involution $(\mathsf{C}, \delta, \eta)$. Given an object $x \in \mathsf{C}$, a **bilinear form** on x is a morphism

$$b : x \longrightarrow (\delta x)^{op}, \tag{10}$$

where $(\delta x)^{op}$ denotes the object δx *viewed as an object in* C. Note that $(\delta x)^{op} = \delta^{op} x^{op}$. Later we will often omit the superscript "op" on objects, leaving the reader to infer, from the context, which category a given object lives in. For the moment we include this notation for extra clarity.

[1] The notation of (1) means that $\delta \dashv \delta^{op}$, i.e. δ is the left adjoint and δ^{op} is the right adjoint.

We call a bilinear form (10) **symmetric** if the following diagram commutes,

$$\begin{array}{ccc} x & \xrightarrow{b} & (\delta x)^{op} \\ & \underset{\eta_x}{\searrow} & \uparrow{\scriptstyle (\delta b)^{op}} \\ & & \delta^{op}\delta x \end{array} \tag{11}$$

where η_x is the component at x of the unit η of the duality involution. Note that $(\delta b)^{op}$ denotes the morphism δb, viewed as a morphism in C. Recall that, by definition, $\hom_{\mathsf{C}^{op}}(z^{op}, y^{op}) = \hom_{\mathsf{C}}(y, z)$; thus, since

$$\delta b : \delta x \to \delta((\delta x)^{op}),$$

$(\delta b)^{op}$ is a morphism with domain $(\delta((\delta x)^{op}))^{op} = \delta^{op}\delta x$ and with codomain $(\delta x)^{op}$.

Given a bilinear form $b : x \to (\delta x)^{op}$, we call the map

$$b^t := (\delta b)^{op} \circ \eta_x \tag{12}$$

the **transpose** of b. This is precisely the composition of the lower path through the diagram (11). Thus a bilinear form b is symmetric if and only if $b = b^t$. In particular, in the example where C is the category of vector spaces (over some fixed field), a bilinear form $b : V \to V^*$ which is symmetric corresponds to a bilinear map (5) for which

$$B(x, y) = B(y, x) \qquad \forall\ x, y \in V. \tag{13}$$

In general, we follow the philosophy that a morphism $b : x \to (\delta x)^{op}$ satisfying (11) is a kind of **fixed point** of the duality involution: morally, x and δx are “the same” up to the data of the morphism b, and we view (11) as a coherence condition. Thus, pairs (x, b) satisfying (11) will be called fixed points.

Skew-symmetric bilinear forms, in the traditional sense (7), may also be encoded as fixed points. Working with the category $\mathsf{vect}_{\mathbf{k}}$, we may take the unit of our duality involution to be given in components by the *negatives* of the maps (4). Then the condition (11) means that b corresponds to a skew-symmetric form in the sense of (7). In particular, we will treat **symplectic structures**, which are non-degenerate, skew-symmetric bilinear forms, in this manner. Questions of linear algebra related to symplectic geometry are the main focus of Part 2 of this thesis.

The study of bilinear forms in the above category-theoretic sense goes back (to the best of the author’s knowledge) to the work of Scharlau and collaborators [**QSSS76**] [**QSS79**] [**Sch75**] in the 1970’s, as well as to work of Serheichuk [**Ser87**] from 1987. Other references which treat bilinear forms (and related structures) in a similar style include [**Jac12**] [**Shi12**] [**FH16**].

The frameworks of Scharlau et al. and Sergeichuk are focused on settings where we have an additive category equipped with an additive duality involution. Their results give tools for solving the following type of problem. Suppose we are given an additive category C in which every object decomposes in an essentially unique way as the direct sum of indecomposable objects, i.e. a version of the Krull-Schmidt theorem holds. If C is equipped with an additive duality involution, then the fixed points form a category of their own, and correspond to objects of C equipped with additional geometric structure. For example, if C is the category of finite-dimensional vector spaces, equipped with the duality involution defined by (2) and (4), then fixed points are vector spaces equipped with a symmetric bilinear form, and morphisms are isometries. A typical geometric classification problem is to classify such fixed points up to isometry. A first step is to classify indecomposable fixed points.

A minimum of necessary material for the Scharlau/Sergeichuk approach is developed in Chapter 4 on additive categories with duality, and in Chapter 6 we apply this to the examples of classifying indecomposable linear Hamiltonian vector fields and classifying indecomposable linear symplectomorphisms (in each case "up to isometry"). Although these classification results are well-known in the symplectic geometry literature (at least over the complex and real number fields), the analysis via duality involutions gives certain conceptual insights of its own. We also apply the Scharlau/Sergeichuk approach in studying symplectic poset representations, which are a major protagonist of Part 2.

The mention of poset representations brings us to the other side of the "loose theme" of this thesis, namely to interactions between duality involutions and representation theory. The representation theoretic thread continues further in Part 3, where we consider Morita bicategories of algebras and the representation pseudofunctor which assigns to each algebra its category of representations.

But let us first spend a moment with poset representations. These are useful in that they serve to encode, in a single formalism, various classification problems in linear algebra, upon which one may then often bring general representation-theoretic results to bear. To use these techniques for linear symplectic geometry, we develop the theory of *symplectic* poset representations in Chapter 7, and we apply this in Chapter 8 for the classification of triples of isotropic subspaces in symplectic vector spaces. This material is essentially that of the paper [**HLW19**].

An "ordinary" **representation of a finite poset** P on a (finite-dimensional) vector space V is an order-preserving map

$$\psi : P \longrightarrow \mathrm{Sub}(V) \tag{14}$$

from P to the poset of linear subspaces of V, ordered by inclusion. To move to a symplectic-geometric setting, we equip P with an order-reversing involution "$(-)^\perp$", and we equip V with a symplectic form ω. Then, a **symplectic poset representation** of $(P, (-)^\perp)$ on (V, ω) is a poset representation ψ of P on V which satisfies the relation

$$\psi(x^\perp) = \psi(x)^\perp \tag{15}$$

where on the right-hand side "$(-)^\perp$" denotes the operation of taking the symplectic orthogonal subspace. That is, if $U \subseteq V$, then $U^\perp = \{v \in V \mid \omega(v, u) = 0 \ \forall u \in U\}$. If we think of the posets P and $\mathrm{Sub}(V)$ as categories, then a representation (14) is nothing but a functor. The two operations "$(-)^\perp$" are (very simple) examples of duality involutions, and the condition (15) is an equivariance condition.

We can also shift a level and consider the category $\mathrm{Rep}(P)$ of *all* representations of a fixed poset P. If P is equipped with an order-reversing involution $(-)^\perp$, we obtain an induced duality involution on $\mathrm{Rep}(P)$ by defining the dual of a representation ψ on V to be the representation ψ^* on V^* given by

$$\psi^*(x) := \psi(x^\perp)^\circ \tag{16}$$

where $(-)^\circ$ denotes the operation of taking the annihilator of a subspace. That is, if $U \subseteq V$, then $U^\circ = \{\xi \in V^* \mid \xi(u) = 0 \ \forall u \in U\}$. Using this duality involution on $\mathrm{Rep}(P)$, symplectic poset representations may be encoded as fixed-points.

In the third, and last, part of this thesis we work with a notion of duality involution which is a "lift" or "categorification" of the definition (1) from the level of "ordinary" categories to the level of bicategories. The definition of a (weak) duality involution on a bicategory was given by M. Schulman in the paper [**Shu18**], in which he also proves a strictification theorem for such structures. While an "ordinary" duality involution involves a functor $\delta : \mathsf{C} \to \mathsf{C}^{op}$ from a category C to its opposite, a **weak duality involution** on a bicategory B involves a pseudofunctor $(-)^{\circ} : \mathsf{B} \to \mathsf{B}^{co}$, where B^{co} denotes the bicategory obtained from B by reversing the direction of only the 2-cells. B^{co} is just one possible generalization of the notion of the opposite category: one may also consider reversing only the 1-cells of B, or reversing *both* 1-cells and 2-cells. The rationale for working with B^{co} is that the notion of weak duality involution is meant to generalize the operation of taking the opposite of a category, which is an example of a duality involution on the bicategory of categories.

We note that "taking the opposite" is a **strict** duality involution in the sense that taking the opposite twice gives back the original category on the nose. In [**Shu18**], Shulman proves a strictification theorem which states that any bicategory with weak duality involution is biequivalent equivariantly to a bicategory with strict duality involution.

In Part 3, which is essentially a reproduction of the paper [**LV19**], we study the bicategory Alg_2 of finite-dimensional $\mathbf{k}$-algebras and their bimodules. Our main result is that the representation pseudofunctor Rep which assigns to each $\mathbf{k}$-algebra its category of representations is in fact equivariant in a natural manner with respect to weak duality involutions which we define on its source and target. Specifically, as source category we take the fully dualisable part of Alg_2, whose objects are *semi-simple* finite-dimensional $\mathbf{k}$-algebras, and as target category we take the fully dualisable part of the bicategory of $\mathbf{k}$-linear categories and right-exact functors. On the latter, a strict duality involution is defined by taking opposite categories, while on $\mathbf{k}$-algebras the duality involution we use is slightly more involved, and its construction is part of our results. A nice feature of this duality involution is that the data required is supplied wholly by the coherence data coming from the definition of Alg_2, and is in this sense "natural". That Rep is equivariant may be interpreted as saying that it gives an explicit strictification of this duality involution on Alg_2.

In a last chapter of the thesis, we sketch a generalisation of our results on the representation pseudofunctor. Since $\mathbf{k}$-algebras are monoids in the category of $\mathbf{k}$-vector spaces, we generalize Alg_2 by considering a bicategory whose objects are monoids in a symmetric fusion category, and we define a representatation pseudofunctor which sends such a monoid to an associated module category.

One shortcoming of this thesis is the absence of "geometry" in the bicategorical setting. Ideally there would be a part defining a notion of fixed point for duality involutions on bicategories, which would be an analogue of the fixed point notion encoding bilinear forms above. As a special case of this bicategorical definition we should recover the 1-categorical notion of duality involution on a category, i.e. we would have that (1) is a bicategorical fixed point of the duality involution of "taking the opposite category". More generally, we would obtain a definition of what it means for an object of a bicategory be equipped with a duality involution. In keeping with the "microcosm principle" [**BD98**], equipping an object with this structure is possible when the ambient bicategory carries a weak duality involution. Tying things together in this way would be beautiful; we must however leave this to future work.

CHAPTER 1

Category theory preliminaries

We assume of the reader a certain familiarity with category theory. Nevertheless, we review some fundamental definitions in order to fix notation and to provide a reminder on aspects which play a role in the following. We also give a quick introduction to topics which are beyond basic category theory, in particular regarding bicategories and enriched categories.

1.1. Basic notions

We recall that a category C is specified by the following data:

- a class of objects $\mathrm{ob}(\mathsf{C})$;
- a class of morphisms $\mathrm{Hom}_{\mathsf{C}}(x, y)$ for every pair of objects (x, y);
- a specified identity morphism $1_x \in \mathrm{Hom}_{\mathsf{C}}(x, x)$ for every object x;
- a composition operation $\circ_{x,y,z} : \mathrm{Hom}_{\mathsf{C}}(y, z) \times \mathrm{Hom}_{\mathsf{C}}(x, y) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(x, z)$ for every triple of objects (x, y, z)

and subject to axioms which express that the composition operations work together associatively, and such that the identity morphisms act neutrally for composition.

The word "class" is used to emphasize that in category theory there are foundational, set-theoretic subtleties involved, due to the fact that, often enough, the objects of a category which "appears in nature" (such as the category Set of all sets) may not form a "set". The meaning of the word "set" here depends on one's choice of axiomatic foundations; to my knowledge however, there are currently no axiomatics which allow one to avoid "size issues", i.e. the fact that some collections of objects are so big that they do not behave as "sets". In the following we will, on the whole, brush over such set-theoretic aspects; we tacitly assume that some choice of axiomatics has been made which gives us a notion of "set" and "class" (whereby a set is a distinguished kind of class, i.e. more axioms hold for sets than for classes), and we will often even forget this distinction. If we use the word "collection", it is meant as a synonym for "class".

In general, we adhere to the convention that the adjective "small" is used to indicate when some class is in fact a set. For example, we say that a category C is **small** if $\mathrm{ob}(\mathsf{C})$ is a set; and a category is **essentially small** if it is equivalent to a small category. A category C is called **locally small** if, for each pair of objects, $\mathrm{Hom}_{\mathsf{C}}(x, y)$ is a set. As used in the previous sentence, a property of a category C holds "locally" if it holds for each $\mathrm{Hom}_{\mathsf{C}}(x, y)$. Sometime we refer to the $\mathrm{Hom}_{\mathsf{C}}(x, y)$ as "hom-sets", even if they might not always technically be "sets". To denote hom-sets, we also use the notation $\mathsf{C}(x, y)$ or $\mathrm{Hom}(x, y)$, assuming, in the latter case, that the category in question is clear. To denote objects, we sometimes write $x \in \mathsf{C}$ instead of $x \in \mathrm{ob}(C)$.

1.1.1. Oppositities. We recall a few details about the operation of taking the opposite of a category, functor, or natural transformation. Opposites play an important role in the first parts of this thesis, and these operations form a key motivating example for the bicategorical definition of a duality involution, which underlies the latter parts.

Definition 1.1.1. *Let C be a category, and let* $\mathrm{ob}(\mathsf{C})$ *be its class of objects. The* ***opposite category*** *of C, usually denoted C^{op}, is the category whose class of objects is the same as that of C, i.e.* $\mathrm{ob}(\mathsf{C}^{op}) := \mathrm{ob}(\mathsf{C})$*, whose morphisms are defined by*

$$\mathrm{Hom}_{\mathsf{C}^{op}}(x, y) := \mathrm{Hom}_{\mathsf{C}}(y, x), \tag{17}$$

and whose composition operations

$$\circ^{\mathsf{C}^{op}}_{x,y,z} : \mathrm{Hom}_{\mathsf{C}^{op}}(y, z) \times \mathrm{Hom}_{\mathsf{C}^{op}}(x, y) \longrightarrow \mathrm{Hom}_{\mathsf{C}^{op}}(x, z), \tag{18}$$

are defined by

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathsf{C}^{op}}(y, z) \times \mathrm{Hom}_{\mathsf{C}^{op}}(x, y) & & & & \mathrm{Hom}_{\mathsf{C}^{op}}(x, z) \\
\| & & & & \| \\
\mathrm{Hom}_{\mathsf{C}}(z, y) \times \mathrm{Hom}_{\mathsf{C}}(y, x) & \xrightarrow{\sigma} & \mathrm{Hom}_{\mathsf{C}}(y, x) \times \mathrm{Hom}_{\mathsf{C}}(z, y) & \xrightarrow{\circ^{\mathsf{C}}_{z,y,x}} & \mathrm{Hom}_{\mathsf{C}}(z, x)
\end{array}$$

where "σ" is the symmetry map associated to the cartesian product.

Remark 1.1.2. From the above definition it is easily seen that $(\mathsf{C}^{op})^{op} = \mathsf{C}$. In particular,

$$\circ^{(\mathsf{C}^{op})^{op}}_{x,y,z} = [(\circ^{\mathsf{C}}_{x,y,z}) \circ \sigma] \circ \sigma = (\circ^{\mathsf{C}}_{x,y,z}) \circ [\sigma \circ \sigma] = \circ^{\mathsf{C}}_{x,y,z}, \tag{19}$$

abusing notation slightly by denoting the two different (mutually inverse) symmetry maps by the same symbol "σ".

Notation 1.1.3. We will use the notation $\mathsf{C}^\circ = \mathsf{C}^{op}$ for the opposite of a category, in order to de-clutter diagrams and formulas. We will also use this notation for the opposites of functors and natural transformations.

Furthermore, it will sometimes be helpful to keep track of whether we are viewing a given object as an element of $\mathrm{ob}(\mathsf{C})$ or as an element of $\mathrm{ob}(\mathsf{C}^\circ)$, even though, by definition, these two classes are "identical". To do this, given $x \in \mathrm{ob}(\mathsf{C})$, we write x° to indicate the corresponding object in $\mathrm{ob}(\mathsf{C}^\circ)$, and vice versa, i.e. if $y \in \mathrm{ob}(\mathsf{C}^\circ)$, then $y^\circ \in \mathrm{ob}(\mathsf{C})$. In particular, $(x^\circ)^\circ = x$ for any object x in either C or C°. We also use this same notation for morphisms: given $f \in \mathrm{Hom}_{\mathsf{C}}(x, y)$, the corresponding morphism in $\mathrm{Hom}_{\mathsf{C}^\circ}(y^\circ, x^\circ)$ is denoted f°.

Definition 1.1.4. *Given a functor $F : \mathsf{C} \to \mathsf{D}$, the* ***opposite functor***

$$F^\circ : \mathsf{C}^\circ \to \mathsf{D}^\circ$$

is defined as follows: given an object $x^\circ \in \mathsf{C}^\circ$,

$$F^\circ(x^\circ) := (F(x))^\circ, \tag{20}$$

and given a morphism $f^\circ : y^\circ \to x^\circ$ in C°,

$$F^\circ(f^\circ) := (F(f))^\circ. \tag{21}$$

Remark 1.1.5. The above definition implies that $(F^\circ)^\circ = F$.

Definition 1.1.6. *Let $F, G : \mathsf{C} \to \mathsf{D}$ be functors, and*

$$\mathsf{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathsf{D}, \qquad \alpha : F \Rightarrow G \tag{22}$$

a natural transformation, with components $\{\alpha_x\}_{x \in \mathrm{ob}(\mathsf{C})}$. The ***opposite natural transformation*** *is the natural transformation*

$$\mathsf{C}^\circ \underset{G^\circ}{\overset{F^\circ}{\rightrightarrows}} \mathsf{D}^\circ, \qquad \alpha^\circ : G^\circ \Rightarrow F^\circ \tag{23}$$

defined in components by

$$\alpha^\circ_{x^\circ} := (\alpha_x)^\circ, \qquad x^\circ \in \mathrm{ob}(\mathsf{C}^\circ). \tag{24}$$

1.1.2. Special kinds of objects and morphisms. A morphism $f : x \to y$ in a category C is a **monomorphism** if, for all objects z,

$$f \circ g = f \circ g' \;\Rightarrow g = g' \tag{25}$$

is true for all morphisms $g, g' \in \mathsf{C}(z, x)$. Similarly, f is an **epimorphism** if it has the analogous cancellation property

$$g \circ f = g' \circ f \;\Rightarrow g = g'. \tag{26}$$

Monomorphisms are als called “monos” or “monic”; epimorphisms “epis” or “epic”. An isomorphism is always both an epi and a mono, but the converse statement does not hold in general. The properties of being “monic” or “epic”, respectively, are closed under composition, and they are dual to each other in the sense that if f is monic in C, then f° is epic in C°, and vice versa.

If a composite gf is monic, then f is necessarily monic. If $f : x \to y$ is an equalizer[1]

$$x \xrightarrow{\;f\;} y \underset{h_1}{\overset{h_2}{\rightrightarrows}} z \tag{27}$$

then f is necessarily monic. Dually, if gf is epic, then g is necessarily epic, and if f is a coequalizer, then f is epic. A number of further properties could be mentioned, but we refrain from making a longer list.

Given an object $x \in \mathsf{C}$, the **slice category over** x, denoted C/x, is the category where objects are morphisms in C having x as codomain, and where a morphism t from $f : y \to x$ to $g : z \to x$ is a morphism $t : y \to z$ in C such that

$$\begin{array}{ccc} y & \xrightarrow{\quad t \quad} & z \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & x & \end{array} \tag{28}$$

commutes.

An object $f : y \to x$ of C/x such that f is monic is called a **subobject** of x, and morphisms of subobjects are morphisms in C/x which are monic in C. Sometimes, for convenience, it will happen that we identify a subobject with its isomorphism class.

If f and g in (28) are monic, then in fact t is necessarily also monic (since $gt = f$ is monic). Also note that if $t' : y \to z$ is another morphism satisfying (28), the cancellation

[1] That is, if f is such that in (27) $h_1 f = h_2 f$ holds.

property of the mono g implies that $t = t'$. Thus there is at most one morphism in C/x between any two monomorphisms of C having codomain x. Given monomorphisms f and g as above, we define

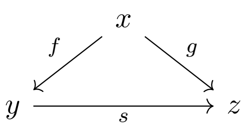
$$f \le g \overset{def}{\Leftrightarrow} \exists\, t \text{ such that (28) holds.} \tag{29}$$

In this manner, the collection of subobjects of x forms a thin[2] category $\mathsf{Sub}(x)$. In particular, if the subobjects of x form a set[3], then $\mathsf{Sub}(x)$ is a preorder. Note that the category $\mathsf{Sub}(x)$ always has a terminal object: the trivial subobject $1_x : x \to x$. We sometimes denote this special subobject by "1" and call it the "top subobject".

Given subobjects $f : u \to x$ and $g : w \to x$, if they have a product then we denote any such by $f \wedge g$, and if f and g have a coproduct then we denote any such by $f \vee g$. Since products and coproducts are unique up to unique isomorphism, we speak of 'the' product or 'the' coproduct. We use similar notation for the product or coproduct of any finite number of subobjects.

Dual to the construction of C/x there is the notion of the **coslice category under** x, which we denote by x/C. Here objects are morphisms in C which have x as their *domain*, and a morphism $s : f \to g$ between two objects of x/C is a morphism s in C such that

(30)



commutes. We define a **quotient object** of x to be an object in x/C which is an epimorphism in C. Comments analogous to the ones above for slice categories and subobjects apply here in dual form.

A **zero object** in C is an object which is both initial and terminal. If $z \in \mathsf{C}$ is a zero object, given any objects x and y, the **zero arrow** $0 : x \to y$ is the composite $x \to z \to y$ of the unique maps into and out of z; it is independent of the choice of zero object z. Note that the unique map from a zero object is always a mono, and the unique map to a zero object is always an epi. Thus, if C has a zero object z, every object x has the subobject $z \to x$, and the quotient x/z. The (unique) zero subobject will be denoted "0" and sometimes called the "bottom subobject".

Assume C has a zero object. Given a morphism $f : x \to y$, a **kernel** of f is an equalizer of $x \underset{0}{\overset{f}{\rightrightarrows}} y$, i.e. an object k and a morphism $i : k \to x$ such that $f \circ i = 0$, and such that for any other morphism $i' : k' \to x$ with $f \circ i' = 0$, there exists a unique $h : k' \to k$ with $k' = kh$. In particular, a kernel of f is unique up to isomorphism in C/x. As an equalizer, a kernel of f is monic, so in particular it defines a subobject of x.

The notion of **cokernel** of a morphism $f : x \to y$ is dual to the notion of kernel: it is a coequalizer of $x \underset{0}{\overset{f}{\rightrightarrows}} y$. As a coequalizer, a cokernel is in particular epic, and hence defines a quotient object.

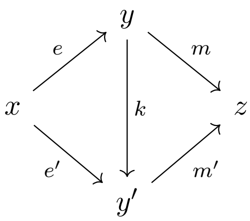
Given a morphism $f : x \to z$ in C, a **factorization** of f is an object y and morphisms $e : x \to y$ and $m : y \to z$ such that $f = me$. Given two factorizations (m, y, e) and

[2]A category is **thin** if for any two objects x and y there is at most one morphism $x \to y$. The intuition is that, in this case, the category is like a preordered set, except that the collection of objects may not be a set.

[3]If this is the case for every object x, then the category C is called "well-powered".

(m', y', e') of f, a morphism $(m, y, e) \to (m', y', e')$ is a morphism $k : y \to y'$ in C such that

(31)



commutes. Factorisations of f form a category $\mathsf{fact}(f)$.

Consider the full subcategory $\mathsf{fact}_{\mathcal{M}}(f)$ of factorizations (m, y, e) of f such that m is monic. The **image** of $f : x \to z$, if it exists, is a factorization (e, y, m) of f which is an initial object in $\mathsf{fact}_{\mathcal{M}}(f)$. In particular, the image of a morphism f defines a subobject of the target of f. The **coimage** of f is the dual notion, i.e. it is a terminal object in the subcategory $\mathsf{fact}_{\mathcal{E}}(f)$ of factorizations (m, y, e) of f such that e is an epimorphism.

A **factorization system** on a category C is a pair of subclasses $(\mathcal{L}, \mathcal{R})$ of the class of morphisms of C such that

- $\mathcal{L}$ and $\mathcal{R}$ are closed under composition, and each contain all isomorphisms;
- each morphism f in C has a factorization $f = me$, with $m \in \mathcal{L}$ and $e \in \mathcal{R}$, and such factorizations are unique up to unique isomorphism.

The archetypical example of a factorization system is the one given in Set by taking $\mathcal{L}$ to be the class of injective functions (the monos), and $\mathcal{R}$ the class of surjective functions (the epis).

1.2. Monoidal categories

Beyond "ordinary" categories, we will also work with categories which are equipped with additional structure. An important example of this is the notion of monoidal category, which is a category equipped with a "product" operation, usually called the monoidal product or tensor product.

Definition 1.2.1. *A* ***monoidal category*** *is specified by the following data, subject to certain axioms:*

- *a category C;*
- *a functor $\otimes : \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C}$, called the* ***monoidal product*** *(we use infix notation $x \otimes y$ and $f \otimes g$, for objects x, y and morphisms f, g);*
- *a* ***monoidal unit*** *$I \in Ob(\mathsf{C})$;*
- *coherence data: natural isomorphisms α, λ, ρ (called* ***associator***, ***left-unitor***, ***right-unitor****), whose respective components are morphisms*

$$\alpha_{x,y,z} : (x \otimes y) \otimes z \longrightarrow x \otimes (y \otimes z),$$
$$\lambda_x : I \otimes x \longrightarrow x,$$
$$\rho_x : x \otimes I \longrightarrow x.$$

The axioms required to hold are the commutativity of two families of diagrams, one of pentagonal form and another of triangular form. The former encodes compatibility between left and right unitors and the associator, while the latter encodes a coherence property of the associator.

We may equip monoidal categories with further structures and/or require further properties.

DEFINITION 1.2.2. *A **braided monoidal category** is a monoidal category $(\mathsf{C}, \otimes, 1)$ equipped with a natural isomorphism β whose components are morphisms*

$$\beta_{x,y} : x \otimes y \longrightarrow y \otimes x. \tag{32}$$

*The **braiding** β is required to satisfy an axiom encoded in hexagonal diagrams which ensure compatibility with the underlying monoidal cagtegory.*

DEFINITION 1.2.3. *A **symmetric monoidal category** is a braided monoidal category $(\mathsf{C}, \otimes, 1, \beta)$ such that*

$$\beta_{x,y} \circ \beta_{x,y} = 1_{x \otimes y} \qquad \forall\, x, y \in Ob(\mathsf{C}). \tag{33}$$

1.3. Bicategories

Loosely speaking, a bicategory is a higher-dimensional analogue of the notion of an "ordinary" category. We may think of an ordinary category as being built from "objects" and "morphisms", the former being "0-dimensional" and the latter being "1-dimensional". In this sense, ordinary categories have two layers: one layer of 0-dimensional constituents and one layer of 1-dimensional constituents.

Bicategories have three layers of structure, made up of 0-dimensional, 1-dimensional, and 2-dimensional constituents, respectively. We call these constituents 0-cells, 1-cells, and 2-cells (though sometimes they are called objects, 1-morphisms and 2-morphisms). In an ordinary category, the morphisms are organized into "hom-sets", i.e. we have a class $\mathrm{Hom}(x, y)$ for every ordered pair of objects (x, y), and for every triple (x, y, z) of objects there is an operation

$$\mathrm{Hom}(y, z) \times \mathrm{Hom}(x, y) \longrightarrow \mathrm{Hom}(x, z)$$

for composing morphisms. The situation is similar for bicategories, but with "hom-categories" in place of hom-sets.

Another remark that is helpful in parsing the definition below is that a monoidal category may be viewed as a very special case of a bicategory, namely one which has only one 0-cell. From this perspective, the monoidal product generalizes to the the composition operation of a bicategory. Also the unit object and coherence data for monoidal categories have generalizations in the definition below.

DEFINITION 1.3.1. *A **bicategory** C is specified by the following data, subject to certain axioms:*

- *a collection* $\mathrm{ob}(\mathsf{C})$ *of 0-cells;*
- *a category $\mathsf{C}(x, y)$ for every ordered pair (x, y) of 0-cells (the objects of these categories are the 1-cells and the morphisms are the 2-cells of C. We call $\mathsf{C}(x, y)$ a "hom-category");*
- *a composition functor*

$$c_{x,y,z} : \mathsf{C}(y, z) \times \mathsf{C}(x, y) \longrightarrow \mathsf{C}(x, z)$$

for every ordered triple of 0-cells (x, y, z). For the composition of 1-cells, we use the notation

$$c_{x,y,z}(g, f) = g \circ f = gf \tag{34}$$

and for the composition of 2-cells,

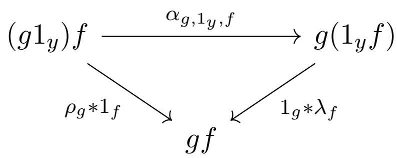
$$c_{x,y,z}(\beta, \alpha) = \beta \star \alpha. \tag{35}$$

The composition (35) of 2-cells is usually called "horizontal composition" of 2-cells in order to distinguish it from a different sort of composition of 2-cells which is possible: namely the composition given within *a given hom-category* $\mathsf{C}(x, y)$ *(this composition is called "vertical composition" of 2-cells). See (36) and (37) below for illustrations of these two types of composition of 2-cells.*

- *a unit functor* $I_x : \mathbf{1} \longrightarrow \mathsf{C}(x, x)$ *for every 0-cell* x*, where* $\mathbf{1}$ *denotes the category which has a single object and a single morphism. Effectively, the unit functor picks out, for each 0-cell* x*, a "unit" 1-cell* 1_x *in* $\mathsf{C}(x, x)$*.*
- *coherence data:*
  - *for each triple* (h, g, f) *of composable 1-cells, an invertible "associator" 2-cell* $\alpha_{h,g,f} : (hg)f \longrightarrow h(gf)$*;*
  - *for each 1-cell* f*, a "left unitor" invertible 2-cell* $\lambda_f : 1_A f \longrightarrow f$*;*
  - *for each 1-cell* f*, a "right unitor" invertible 2-cell* $\rho : f1_A \longrightarrow f$*.*

The axioms that this data must satisfy are, briefly, that

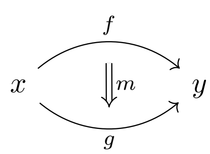
- *the associator 2-cells must assemble into a natural isomorphism, and similarly so for the left- and right-unitor 2-cells, respectively;*
- *the diagram*



  must commute for all 1-cells $g \in \mathsf{C}(y, z)$*,* $f \in \mathsf{C}(x, y)$*;*
- *the associator 2-cells must satisfy a compatibility with the unit 1-cells which is encoded as the commutativity of a certain "pentagram diagram" (see* [**Lei98**] *for the full diagram).*

Remark 1.3.2. If the coherence data in the above definition are all identities, then the axioms hold automatically, and the bicategory in question is called a **strict bicategory** or a **2-category**.

Remark 1.3.3. Often we use "globular" diagrams to visualize situations in bicategories: if x and y are 0-cells, f and g 1-cells from x to y, and m a 2-cell from f to g, then we draw this as so:



Horizontal and vertical composition of 2-cells may be illustrated, respectively, as follows:

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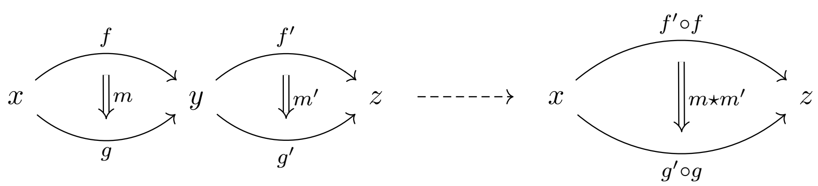

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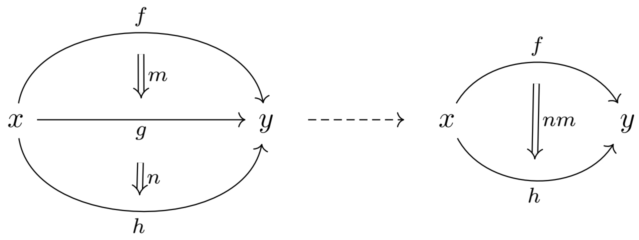
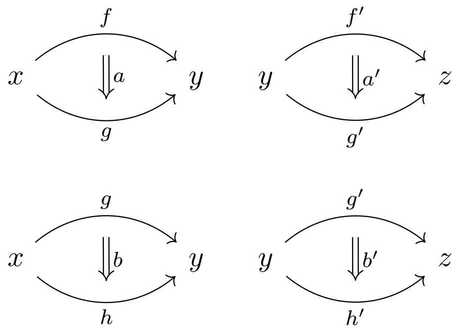


Remark 1.3.4. Let C be a bicategory, let $x, y, z \in \mathsf{C}$ be 0-cells, let $f, g, h : x \to y$ and $f', g', h' : y \to z$ be 1-cells, and let $a : f \Rightarrow g$, $b : g \Rightarrow h$, $a' : f' \Rightarrow g'$, and $b' : g' \Rightarrow h'$ be 2-cells, as such:



Then the functoriality of the composition functor $c_{x,y,z}$ of C implies in particular that

$$(b \circ_{\mathsf{C}(y,z)} b') \star (a \circ_{\mathsf{C}(x,y)} a') = (b \star b') \circ_{\mathsf{C}(x,z)} (a \star a'). \tag{38}$$

In other words, in the globular diagram above, it does not matter if we first horizontally compose along the top and bottom rows respectively, and then vertically compose, or if we first vertically compose along the left and right columns respectively, and then horizontally compose. The equation (38) is often called the "**interchange law**".

Example 1.3.5. There is a bicategory CAT which has the following data:

- 0-cells are categories;
- 1-cells are functors between categories;
- 2-cells are natural transformations between functors.

Composition of 1-cells is the usual composition of functors. Horizontal composition of 2-cells is defined as follows: given 2-cells α and β as such

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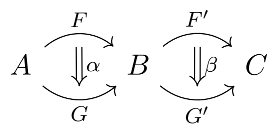
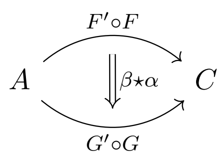


the horizontal composition $\beta \star \alpha$ is the natural transformation

(40)


defined in components by

$$(\beta \star \alpha)_x := \beta_{G(x)} \circ F'(\alpha_x)$$

for $x \in \mathrm{Ob}(A)$. $\triangle$

REMARK 1.3.6. In the previous example, if in a horizontal composition $\beta \star \alpha$ we have $F = G$ and $\alpha = \mathrm{id}_F$, then sometimes this is written $\beta \star F$ instead of $\beta \star \mathrm{id}_F$. The resulting composite natural transformation is called the "left-whiskering" of β by F. There is also an analogous notion of "right-whiskering". The name comes from the fact that in the globular diagram (36), if $F = G$ and $\alpha = \mathrm{id}_F$, then we may replace the arrows in the left-hand "globule" by a single horizontal arrow for F; this horizontal arrow is like a whisker attached to the left side of the other globule in the diagram.

We use the notation Cat to denote the bicategory defined as above, but such that 0-cells are only locally small categories; similarly, we use cat when only considering small categories.

EXAMPLE 1.3.7. There is a strict bicategory linRel which has the following data:

- 0-cells are finite-dimensional vector spaces over a fixed ground field $\mathbf{k}$;
- 1-cells are linear relations between vector spaces;
- 2-cells are inclusion relations between linear relations: in other words, given linear relations $f : V \to W$ and $g : V \to W$, there is a (single, unique) 2-cell $f \to g$ if and only if $f \subseteq g$.

△

Just as ordinary categories assemble naturally into the "2-dimensional" structure of a bicategory (which has three levels of stucture)[4], there are also "3-dimensional" structures which describe the totality of bicategories. The corresponding four levels of structure for bicategories are:

- 0-cells: bicategories;
- 1-cells: bifunctors;
- 2-cells: transformations between bifunctors;
- 3-cells: modifications between transformations.

We briefly discuss these notions, as these will be used in the last part of the thesis. In particular, we wish to fix terminologies, since these are not always consistent in the literature. For further details regarding bicategories, we refer to [**Lei98**], which is a clear and concise exposition of the essential definitions.

DEFINITION 1.3.8. *Let C and D be bicategories. A* ***bifunctor*** $F : \mathsf{C} \longrightarrow \mathsf{D}$ *is defined by the following data, subject to certain axioms:*

- *a function $F : Ob(\mathsf{C}) \longrightarrow Ob(\mathsf{D})$;*
- *a functor $F_{x,y} : \mathsf{C}(x,y) \longrightarrow \mathsf{D}(Fx, Fy)$ for every pair of 0-cells (x,y) of C;*
- *coherence data:*
  - *for every 0-cell x of C, a 2-cell $\phi^F_x : 1_{Fx} \Longrightarrow F1_x$ in D which "witnesses" the preservation of units by F;*
  - *for every pair (g,f) of composable 1-cells in C, a 2-cell $\phi^F_{g,f} : Fg \circ Ff \Longrightarrow F(g \circ f)$ in D which "witnesses" the compositionality of F.*

The axioms required for this data are encoded in three commutative diagrams which encode compatibilities between the data of F and the bicategories C and D; c.f. [**Lei98**].

[4] C.f. Example 1.3.5.

Given bifunctors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{E}$, their composition GF is given as follows: on the level of objects it is the composition of functions; on the level of morphism-categories it is the composition of functors; and the coherence data is

- $\phi^{GF}_x := G(\phi^F_x) \circ \phi^G_{Fx}$;
- $\phi^{GF}_{g,f} := G(\phi^F_{g,f}) \circ \phi^G_{Fg,Gf}$.

REMARK 1.3.9. If, in the above definition, the coherence data are given by invertible 2-cells, then we speak of a **strong bifunctor** or, synonymously, of a **pseudofunctor**. If in fact all the coherence 2-cells are equalities, then we speak of a **strict bifunctor**. Thus, our "baseline notion" of morphism of bicategories is the "weakest" notion (this is what we call a bifunctor), and all other variants are "stronger". This corresponds to the approach taken in [**Lei98**]. Other authors however, e.g. [**Gur12**], take their baseline notion of morphism of bicategories to be what we here are calling a strong bifunctor or a pseudofunctor (and they variously call this notion a "functor", a "weak functor", or a "pseudofunctor"). In their cases, the adjective "lax" is added to indicate the weaker notion (i.e. what we call a bifunctor), and the adjective "strict" is added to indicate the stronger notion (i.e. what we call a strict bifunctor).

DEFINITION 1.3.10. *Let $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{C} \to \mathsf{D}$ be pseudofunctors between bicategories. A **transformation** $\tau : F \to G$ is defined by the following data, subject to certain axioms:*

- *for every 0-cell $x \in \mathsf{C}$, a 1-cell $\tau_x : Fx \to Gx$ in D;*
- *for every 1-cell $f : x \to y$ in C, a 2-cell $\tau_f : Gf \circ \tau_x \Rightarrow \tau_y \circ Ff$.*

The axioms required are encoded in two commutative diagrams; c.f. [**Lei98**].

Given transformations $\tau : F \to G$ and $\pi : G \to H$, their composition $\pi\tau$ is defined by

- *$(\pi\tau)_x := \pi_x \circ \tau_x$ for 0-cells $x \in \mathsf{C}$;*
- *$(\pi\tau)_f := (\beta_f \star 1_{\alpha_x}) \circ (1_{\beta_y} \star \alpha_f)$ for 1-cells $f : x \to y$ in C.*

REMARK 1.3.11. If, in the above definition, the 2-cells are invertible, then we speak of a **strong transformation** or, synonymously, of a **pseudonatural transformation**. If in fact all the coherence 2-cells are equalities, then we speak of a **strict transformation**.

DEFINITION 1.3.12. *Let $\tau : F \to G$ and $\kappa : F \to G$ be transformations between pseudofunctors $\mathsf{C} \to \mathsf{D}$. A **modification** $\Gamma : \tau \to \kappa$ is defined by the following data, subject to one axiom:*

- *for every 0-cell x of C, a 2-cell $\Gamma_x : \tau_x \Rightarrow \kappa_x$.*

The axiom required is that the following diagram commutes for every 1-cell $f : x \to y$ in C:

$$\begin{array}{ccc} Gf \circ \tau_x & \xRightarrow{1\star\Gamma_x} & Gf \circ \kappa_x \\ {\scriptstyle\tau_f}\Downarrow & & \Downarrow{\scriptstyle\kappa_f} \\ \tau_y \circ Ff & \xRightarrow[\Gamma_y\star 1]{} & \kappa_x \circ Ff. \end{array} \tag{41}$$

Given modifications $\Gamma : \tau \to \kappa$ and $\Gamma' : \kappa \to \pi$, their composition $\Gamma'\Gamma : \tau \to \pi$ is given in components by $(\Gamma'\Gamma)_x := \Gamma'_x\Gamma_x$, where the right-hand side is simply vertical composition of 2-cells.

REMARK 1.3.13. If all the 2-cell in the above definition are invertible, then the modification Γ is called invertible.

REMARK 1.3.14. Given bicategories C and D there is a bicategory $[\mathsf{C},\mathsf{D}]$ where

- 0-cells are bifunctors $\mathsf{C} \to \mathsf{D}$;
- 1-cells are transformations between pseudofunctors;
- 2-cells are modifications between transformations.

This is analogous to the fact that, given ordinary categories C and D, there is a category $[\mathsf{C},\mathsf{D}]$ where objects are functor $\mathsf{C} \to \mathsf{D}$ and morphisms are natural transformation.

REMARK 1.3.15. There are also variants to $[\mathsf{C},\mathsf{D}]$ from the previous remark. For example, given bicategories C and D there is in fact the bicategory $[\mathsf{C},\mathsf{D}]_{1,0}$, where objects are taken to be strong bifunctors and 1-cells are transformations; and $[\mathsf{C},\mathsf{D}]_{0,1}$, where objects are bifunctors while 1-cells are assumed to be strong transformations; and $[\mathsf{C},\mathsf{D}]_{1,1}$, where both bifunctors and transformations are assumed to be strong. (Our notation is non-standard here). We reserve the notation $[\mathsf{C},\mathsf{D}]$ for the weakest variant.

REMARK 1.3.16. In [**Gur07**], a definition of **tricategory** is given. In a tricategory T, there is in particular a bicategory $\mathsf{T}(x,y)$ for every pair of 0-cells (x,y). In order to have a tricategory Bicat of bicategories, given 0-cells C and D (which are bicategories), one must take $\mathsf{Bicat}(\mathsf{C},\mathsf{D}) = [\mathsf{C},\mathsf{D}]_{1,1}$; the weaker options do not work in the sense that that the definition of a tricategory is not satisfied. This is related to the fact that only if one works with strong bifunctors and strong transformations does there seem to be a clear and well-behaved notion of "horizontal composition of transformations".

To illustrate the previous remark and to lead into the full definition of horizontal composition of transformations, let us first define left- and right-whiskering of a transformation by a bifunctor. That is, we consider the situation

$$\mathsf{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathsf{B} \underset{G'}{\overset{F'}{\rightrightarrows}} \mathsf{C}, \qquad \alpha : F \Rightarrow G, \quad \beta : F' \Rightarrow G' \tag{42}$$

where α and β are transformations, and where either

(1) Left-whiskering: $F = G$ and $\alpha = 1_F$, or
(2) Right-whiskering: $F' = G'$ and $\beta = 1_{F'}$.

In the first case, we define the horizontal composition $\beta \star 1_F$ by

$$(\beta \star 1_F)_x := \beta_{Fx} : F'Fx \longrightarrow G'Fx, \tag{43}$$

for 0-cells $x \in \mathsf{A}$. Given a 1-cell $f : x \to y$ in A, we define the 2-cell

$$(\beta \star 1_F)_f : F'Ff \circ \beta_{Fx} \longrightarrow \beta_{Fy} \circ G'Ff,$$

to be

$$(\beta \star 1_F)_f := \beta_{Ff}. \tag{44}$$

In the second case, to define the horizontal composition $1_{F'} \star \alpha$, let

$$(1_{F'} \star \alpha)_x := F'(\alpha_x) : F'Fx \longrightarrow F'Gx, \tag{45}$$

for 0-cells $x \in \mathsf{C}$. To define

$$(1_{F'} \star \alpha)_f : F'Ff \circ (1_{F'} \star \alpha)_x \Rightarrow (1_{F'} \star \alpha)_y \circ F'Gf,$$

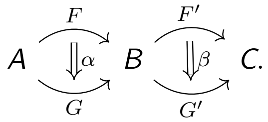
given a 1-cell $f : x \to y$ in A, the natural choice is to let $(1_{F'} \star \alpha)_f$ be the composite

$$F'Ff \circ F'(\alpha_x) \overset{\phi^{F'}_{Ff,\alpha_x}}{\Longrightarrow} F'(Ff \circ \alpha_x) \overset{F'(\alpha_f)}{\Longrightarrow} F'(\alpha_y \circ Gf) \overset{(\phi^{F'}_{\alpha_y,Ff})^{-1}}{\Longrightarrow} F'(\alpha_y) \circ F'Gf, \tag{46}$$

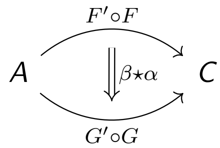
which requires that $\phi^{F'}_{\alpha_y,Ff}$ is invertible.

We keep the above definitions for left- and right-whiskering and use them to define general horizontal composition of transformations between strong bifunctors.

Definition 1.3.17. *Let A, B, C be bicategories, F, F', G, G' strong bifunctors, and α, β strong transformations such that*

(47) 

The horizontal composition $\beta \star \alpha$ is the strong transformation

(48) 

defined as follows.

- *Given a 0-cell x in A, let*

(49) $$(\beta \star \alpha)_x := (\beta \star 1_{F'})_x \circ (1_G \star \alpha)_x = \beta_{F'x} \circ G(\alpha_x);$$

  this is a 1-cell $F'Fx \longrightarrow G'Gx$.

- *Given a 1-cell $f : x \to y$ in A, let*

(50) $$(\beta \star \alpha)_f := ((\beta \star 1_{F'})_f \star 1_{(1_G \star \alpha)_x}) \circ (1_{(\beta \star 1_{F'})_y} \star (1_G \star \alpha)_f)$$

(51) $$= ((\beta_{F'f} \star 1_{G(\alpha_x)}) \circ (1_{\beta_{F'y}} \star ((\phi^G_{\alpha_y,F'f})^{-1} \circ G(\alpha_f) \circ \phi^G_{Ff,\alpha_x}));$$

  this is a 2-cell $F'Ff \circ (\beta \star \alpha)_x \Rightarrow (\beta \star \alpha)_y \circ G'Gf$.

1.4. Adjunctions

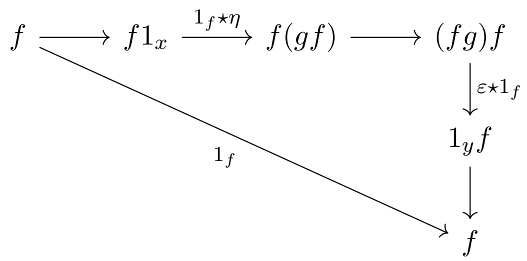
We discuss adjunctions in a bicategory C, following the exposition in [**Gur12**]. The most important special case is, of course, when C is the bicategory of categories, in which case we recover the 'classical' notion of adjoint functors.

Fix a bicategory C, with coherence data α, λ, ρ.

Definition 1.4.1. *Let $f : x \to y$ and $g : y \to x$ be 1-cells in C. An **adjunction** $f \dashv g$ is the data of 2-cells*

$$\eta : 1_x \Rightarrow gf \qquad \text{and} \qquad \varepsilon : fg \Rightarrow 1_y$$

making the following diagrams commute (unlabeled arrows are obvious coherence morphisms):

(52) 

$$\begin{array}{ccccccc} g & \longrightarrow & 1_x g & \xrightarrow{\eta\star 1_g} & (gf)g & \longrightarrow & g(fg) \\ & \searrow^{1_g} & & & & & \downarrow{\scriptstyle 1_g\star\varepsilon} \\ & & & & & & g1_y \\ & & & & & & \downarrow \\ & & & & & & g \end{array} \tag{53}$$

*We say that f is **left-adjoint** to g, and g is **right-adjoint** to f. The 2-cells η and ε are called, respectively, the **unit** and the **co-unit** of the adjunction. We bundle the data of an adjunction as a quadruple $(f, g, \eta, \varepsilon)$.*

*An adjunction is an **adjoint equivalence** if the unit and co-unit are invertible.*

Remark 1.4.2. The identities encoded by (52) and (53) are often called the **triangle identities**.

Remark 1.4.3. If $\mathsf{C} = \mathsf{Cat}$, then in particular the coherence data of C is trivial. Given functors $F : X \to Y$, $G : Y \to X$ and natural transformations $\eta : 1_X \Rightarrow GF$ and $\varepsilon : GF \Rightarrow 1_Y$, in terms of components the diagram (52) reads, for each $x \in \mathrm{Ob}(X)$, as

$$\begin{array}{ccc} F(x) & \xrightarrow{F(\eta_x)} & FGF(x) \\ & \searrow^{1_{F(x)}} & \downarrow{\scriptstyle \varepsilon_{F(x)}} \\ & & F(x) \end{array} \tag{54}$$

and a similar collection of diagrams corresponds to (53).

Proposition 1.4.4. *If $(f, g, \eta, \varepsilon)$ is an adjoint equivalence, then so is $(g, f, \varepsilon^{-1}, \eta^{-1})$.*

Proof. This is Proposition A.1.14 in [**Gur07**]. □

Definition 1.4.5. *A 1-cell $f : x \to y$ is an **equivalence** if there exits a 1-cell $g : y \to x$ such that $gf \simeq 1_x$ and $fg \simeq 1_y$. We call such a 1-cell g a **pseudoinverse** of f.*

Definition 1.4.6. *Let C and D be bicategories, and let $F, G : \mathsf{C} \to \mathsf{D}$ be pseudofunctors. A pseudonatural transformation $\alpha : F \to G$ is called a **pseudonatural isomorphism** if α is an isomorphism between the F and G in the bicategory $[\mathsf{C}, \mathsf{D}]$. The notions of **pseudonatural equivalence**, **pseudonatural adjunction**, and **pseudonatural adjoint equivalence** are defined similarly.*

The following follows from Section 1 in [**Gur12**], see in particular Theorem 1.9 and Remark 1.10 there.

Theorem 1.4.7. *Let $f : x \to y$ be an equivalence in C, let g be a pseudoinverse of f, and let $\alpha : 1_x \Rightarrow gf$ be invertible. Then there exists a unique adjoint equivalence $f \dashv g$ with unit $\eta = \alpha$.*

Beside the notion of an equivalence "internal" to a bicategory, as in Definition 1.4.5, there is also a notion whichmay be considered the appropriate notion of "equivalence *between* bicategories". We include this definition here for later reference.

Definition 1.4.8. *Let C and D be bicategories. A bifunctor $F : \mathsf{C} \to \mathsf{D}$ is a **biequivalence** if there exists a bifunctor $G : \mathsf{C} \to \mathsf{D}$ and transformations $\eta : 1_{\mathsf{C}} \to GF$ and $\varepsilon : FG \to 1_{\mathsf{D}}$ such that η is an equivalence in the bicategory $[\mathsf{C}, \mathsf{C}]$ and ε is an equivalence in the bicategory $[\mathsf{D}, \mathsf{D}]$.*

REMARK 1.4.9. To say that η is an equivalence in the bicategory $[\mathsf{C}, \mathsf{C}]$ means that there is a transformation $\eta^{\square} : GF \to 1_{\mathsf{C}}$ and invertible modifications $\alpha : 1_{1_{\mathsf{C}}} \to \eta^{\square} \circ \eta$ and $\alpha^{\square} : \eta \circ \eta^{\square} \to 1_{GF}$.

Similarly, to say that ε is an equivalence in the bicategory $[\mathsf{D}, \mathsf{D}]$ means that there is a transformation $\varepsilon^{\square} : 1_{\mathsf{D}} \to FG$ and invertible modifications $\beta : 1_{FG} \to \eta^{\square} \circ \eta$ and $\beta^{\square} : \eta \circ \eta^{\square} \to 1_{1_{\mathsf{D}}}$.

1.5. Enriched, additive, and abelian categories

In this section we briefly recall the notion of enriched category, and give some basic definitions related to additive and abelian categories. For the former our main reference is [**Kel05**]; for the latter, see [**ML98**].

1.5.1. Enriched categories. The basic idea is as follows. In an ordinary (locally small) category C, for each pair of objects (x, y), there is a hom-set of morphims $\mathsf{C}(x, y)$, and the collection of all such hom-sets is woven together by composition functions

$$\mathsf{C}(y, z) \times \mathsf{C}(x, y) \longrightarrow \mathsf{C}(x, z).$$

To build an enriched category C, we choose some monoidal category V, and we replace the hom-sets with "*hom-objects*" $\mathsf{C}(x, y) \in \mathsf{V}$ (one for each pair of objects (x, y) of C), and the collection of all these "hom-objects" is linked together by composition *morphisms* (morphisms in V)

$$\mathsf{C}(y, z) \otimes \mathsf{C}(x, y) \longrightarrow \mathsf{C}(x, z),$$

where "$\otimes$" here is the monoidal product in V. The resulting structure is called a **V-enriched category** or a **V-category**.

In an ordinary category C, each hom-set $\mathsf{C}(x, x)$ has an identity *element*; in a V-category C, each hom-object $\mathsf{C}(x, x) \in \mathsf{V}$ has an identity "element" encoded as a morphism $I \longrightarrow \mathsf{C}(x, x)$, where I is the monoidal unit of V.

The remainder of the formal definition of a V-category consists of axioms which encode the associativity of the composition operations and the unitality of the identity "elements"; we omit the full details and refer to [**Kel05**].

EXAMPLE 1.5.1. Consider the monoidal category $\mathsf{V} = (\mathsf{Set}, \times, \{*\})$. A V-category then amounts exactly to an ordinary (locally small) category. In this sense, the theory of enriched categories generalizes the theory of ordinary categories.

EXAMPLE 1.5.2. Let $\mathsf{V} = (\mathsf{Cat}, \times, I)$, where Cat is the category of categories and I is the category with only one morphism. Then a V-category is the same thing as a 2-category.

REMARK 1.5.3. If the enriching category V has a forgetful functor $U : \mathsf{V} \longrightarrow \mathsf{Set}$, we may think of a V-category C as having an "underlying" ordinary category which has hom-sets $U\mathsf{C}(x, y)$. In this case, the distinction between C and its "underlying" category, call it C_0, can be important to keep in mind. For instance, an endomorphism of C_0 is any ordinary functor $\mathsf{C}_0 \to \mathsf{C}_0$, while an endomorphism of C, as a V-category, must, by definition, be a V-functor (see the definition below). Sometimes, however, it is convenient to blur the distinction between C and C_0, and to think of C rather as an ordinary category equipped with extra structure.

Given V-categories C and D, a **V-functor** $F : \mathsf{C} \longrightarrow \mathsf{D}$ consists of a function

$$\mathrm{ob}(\mathsf{C}) \longrightarrow \mathrm{ob}(\mathsf{D})$$

and a collection of morphisms

$$F_{x,y} : \mathsf{C}(x, y) \longrightarrow \mathsf{D}(Fx, Fy)$$

for each pair of objects (x, y) in C, satisfying the usual kind of compatibility with the composition operations and identity morphisms.

Given V-functors $F, G : \mathsf{C} \longrightarrow \mathsf{D}$, a V-**natural transformation** $\alpha : F \Rightarrow G$ consists of a collection of **components**

$$\alpha_x : I \longrightarrow \mathsf{D}(Fx, Gx) \qquad x \in \mathrm{ob}(\mathsf{C})$$

which satisfy the commutativity of diagrams which generalize the usual naturality squares for ordinary natural transformations.

V-categories, V-functors, and V-natural transformations assemble into a 2-category V-Cat of V-enriched categories. We also note that given V-categories C and D, and assuming that C is essentially small, the category of V-functors $[\mathsf{C}, \mathsf{D}]_{\mathsf{V}}$ is itself a V-category in a natural manner.

In the following, we will be interested in enriching categories V which are *symmetric monoidal*, because for such V we may define the *opposite* C° of a V-category, and this will again be a V-category. To see how this works, let $\mathsf{V} = (\mathsf{V}, \otimes, I, \sigma)$ be symmetric monoidal, and C a V-category. The **opposite** C° of C has the same objects as C, and its hom-objects are defined by

$$\mathsf{C}^\circ(x, y) = \mathsf{C}(y, x) \qquad x, y \in \mathrm{ob}(\mathsf{C}^\circ) = \mathrm{ob}(\mathsf{C}).$$

The composition morphisms for C°

$$\mathsf{C}^\circ(y, z) \otimes \mathsf{C}^\circ(x, y) \longrightarrow \mathsf{C}^\circ(x, z)$$

are defined by

$$\mathsf{C}(z, y) \otimes \mathsf{C}(y, x) \xrightarrow{\sigma} \mathsf{C}(y, x) \otimes \mathsf{C}(z, y) \longrightarrow \mathsf{C}(z, x)$$

(the second arrow being composition in C), and the identity morphisms

$$I \longrightarrow \mathsf{C}^\circ(x, x) = \mathsf{C}(x, x) \qquad x \in \mathrm{ob}\mathsf{C}^\circ = \mathrm{ob}\mathsf{C}.$$

are the same ones as for C.

In a similar fashion as with ordinary categories, we may also define the opposites of V-functors and V-natural transformations. Given a V-functor $F : \mathsf{C} \to \mathsf{D}$, its opposite $F^\circ : \mathsf{C}^\circ \to \mathsf{D}^\circ$ is identical with F on objects, and its component morphisms are defined by $F^\circ_{x,y} := F_{y,x}$. The opposite of a V-natural transformation $\alpha : F \Rightarrow G$ is the V-natural transformation $\alpha^\circ : G^\circ \Rightarrow F^\circ$ defined in components by

$$(\alpha^\circ)_x := \alpha_x : I \longrightarrow \mathsf{D}^\circ(G^\circ x, F^\circ x) = \mathsf{D}(Fx, Gx).$$

Thus, as with ordinary categories, the operation “op” on V-Cat (for V symmetric monoidal) inverts the direction of 2-cells, but not of 1-cells.

1.5.2. Additive categories. Let Ab denote the (symmetric monoidal) category of abelian groups. We consider here a special kind of Ab-enriched category. Since there is a forgetful functor $U : \mathsf{Ab} \longrightarrow \mathsf{Set}$, in the following we often view Ab-enriched categories as ordinary categories which happen to have extra structure (see Remark 1.5.3).

An **additive category** is an Ab-enriched category C which has a zero object (an object which is both initial and terminal) and all binary biproducts.

Recall that a biproduct of a pair of objects (x, y) is an object z, together with morphisms

$$x \underset{i_x}{\overset{p_x}{\rightleftarrows}} z \underset{i_y}{\overset{p_y}{\rightleftarrows}} y \tag{55}$$

such that

$$p_x i_x = 1_x, \quad p_y i_y = 1_y, \quad i_x p_x + i_y p_y = 1_z.$$

These equation imply in particular that i_x, i_y are monic, and p_x, p_y are epic. For a biproduct of (x, y) we use the symbol "$x \oplus y$" and we will often speak of a "direct sum". We may think of $\oplus$ as a functor $\mathsf{C} \times \mathsf{C} \to \mathsf{C}$ if we choose, for each pair of objects, a specific biproduct. The notion of biproduct also extends, in an obvious manner, to any finite number of objects.

REMARK 1.5.4. In an Ab-enriched category, an initial object is automatically also terminal (and vice versa), a (binary) coproduct is automatically also a product (and vice versa), and (co)products are in fact biproducts (see [**ML98**] VIII.2).[5] In particular, biproducts are thus unique up to unique isomorphism.

REMARK 1.5.5. Both $x \oplus y$ and $y \oplus x$ will be "the" (co)product of x and y, so we always have a canonical isomorphism

$$\sigma_{x,y} : x \oplus y \longrightarrow y \oplus x, \tag{56}$$

and there are similar maps for biproducts with more summands. Sometimes we will wish to keep track of the ordering in a biproduct, in which case such "symmetry maps" will be useful.

REMARK 1.5.6. If C is additive, the abelian group structures ("+") on the hom-sets are related to the biproduct operation $\oplus$: given morphisms $f, g : x \longrightarrow y$ in an additive category C,

$$f + g = \nabla \circ (f \oplus g) \circ \Delta$$

where $\Delta : x \to x \times x \simeq x \oplus x$ is the diagonal map (i.e. built from the pairing $\langle 1_x, 1_x \rangle : x \to x \times x$), and $\nabla : y \oplus y \simeq y \coprod y \to y$ the codiagonal (i.e. built from the copairing $[1_y, 1_y] : y \coprod y \to y$).

REMARK 1.5.7. In an additive category, the equalizer of $x \underset{g}{\overset{f}{\rightrightarrows}} y$, if it exists, is $\mathrm{Ker}\,(f - g)$. In particular, if this kernel is zero, then the maps are equal.

REMARK 1.5.8. In an additive category, for any object x, the set $\mathrm{Hom}(x, x) = \mathrm{End}(x)$ naturally carries the structure of a (unital) ring, with addition coming from the Ab-enrichment, and multiplication given by composition. The unit is the identity morphism 1_x. If $n \in \mathbb{N}$ is an integer, the notation $n \in \mathrm{End}(x)$ denotes the n-fold sum of 1_x with itself. We write $1/n \in \mathrm{End}(x)$ if $n \in \mathrm{End}(x)$ is an isomorphism (with inverse denoted by $1/n$). A useful fact is that $\mathrm{End}(x)$ is the zero ring if and only if x is a zero object, and this is the case if and only if $1_x = 0_x$.

[5]Thus an additive category may in fact alternatively be defined as an Ab-enriched category which has all finite coproducts, say.

We define an **additive functor** $F : \mathsf{C} \longrightarrow \mathsf{D}$ between additive categories to be simply an Ab-enriched functor. It turns out that this is equivalent to saying that F is an ordinary functor which preserves biproducts. For any additive functor we have canonical isomorphisms $\varphi^F_{x,y} : F(x \oplus y) \to F(x) \oplus F(y)$ natural in (x, y). The morphism $\varphi^F_{x,y}$ is the one guaranteed by the universal property of $F(x) \oplus F(y)$ as a product: it is the unique morphism such that

$$\begin{array}{ccccc} & & F(x \oplus y) & & \\ & \swarrow^{F(p_x)} & \downarrow{\scriptstyle \varphi^F_{x,y}} & {}^{F(p_y)}\searrow & \\ F(x) & \xleftarrow[p_{F(x)}]{} & F(x) \oplus F(y) & \xrightarrow[p_{F(y)}]{} & F(y) \end{array} \tag{57}$$

commutes. The inverse morphism $(\varphi^F_{x,y})^{-1}$, on the other hand, is the morphism guaranteed by the universal property of $F(x) \oplus F(y)$ as a coproduct: it is the unique morphism such that

$$\begin{array}{ccccc} F(x) & \xrightarrow{i_{F(x)}} & F(x) \oplus F(y) & \xleftarrow{i_{F(y)}} & F(y) \\ & \searrow_{F(i_x)} & \uparrow{\scriptstyle (\varphi^F_{x,y})^{-1}} & {}_{F(i_y)}\swarrow & \\ & & F(x \oplus y) & & \end{array} \tag{58}$$

commutes. These facts are useful when proving various statements involving additive functors and biproducts.

REMARK 1.5.9. For instance, $\varphi^F_{x,y}$ and $\varphi^F_{y,x}$ are related by

$$\varphi^F_{x,y} = \sigma^{-1}_{Fx,Fy} \circ \varphi^F_{y,x} \circ F(\sigma_{x,y}), \tag{59}$$

where the "σ" maps are the respective symmetry isomorphisms for the biproduct. This may be proved by noting that both sides of the equation are morphisms satisfying the universal property of the unique canonical map $F(x \oplus y) \to F(x) \oplus F(x)$.

REMARK 1.5.10. We also note here that additive functors map zero objects to zero objects.

Together with the usual notion of natural transformation, additive categories and additive functors form a sub-2-category of Ab-enriched categories. In other words, a natural transformation between additive functors is defined to be simply an ordinary natural transformation between functors. Although we do not impose any conditions in this definition, natural isomorphisms between additive functors will necessarily satisfy a certain additivity property. Namely, let $F, G : \mathsf{C} \to \mathsf{D}$ be additive functors, with corresponding coherence maps

$$\varphi^F_{x,y} : F(x \oplus y) \to F(x) \oplus F(y) \quad \text{and} \quad \varphi^G_{x,y} : G(x \oplus y) \to G(x) \oplus G(y),$$

and let $\alpha : F \Rightarrow G$ be a natural isomorphism. Then

$$\alpha_x \oplus \alpha_y = \varphi^G_{x,y} \circ \alpha_{x \oplus y} \circ (\varphi^F_{x,y})^{-1} \tag{60}$$

holds, since both $\varphi^G_{x,y}$ and $\alpha_x \oplus \alpha_y \circ \varphi^F_{x,y} \circ \alpha^{-1}_{x \oplus y}$ satisfy a universal property of the coherence map $\varphi^G_{x,y} : G(x \oplus y) \to G(x) \oplus G(y)$. For similar reasons, if $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{E}$ are additive functors, with respective coherence binatural isomorphisms φ^F and φ^G, then the coherence binatural isomorphism φ^{GF} satisfies

$$\varphi^{GF}_{x,y} = \varphi^G_{Fx,Fy} \circ G\varphi^F_{x,y} \tag{61}$$

for every pair of objects (x, y) in C.

REMARK 1.5.11. Several basic constructions used to build new categories from old ones also work well within the "additive setting".

For instance, let D be an additive category, and let C be an essentially small category, not necessarily additive. Then the category $[\mathsf{C}, \mathsf{D}]$ of all functors from C to D is again additive. This is because we may define an "addition operation" component-wise (using the addition in D) for natural transformations having the same source and target. In other words, if $\alpha, \beta : F \Rightarrow G$, then we set

$$(\alpha + \beta)_x := \alpha_x + \beta_x$$

for each object $x \in \mathsf{C}$. This gives an Ab-enrichment on $[\mathsf{C}, \mathsf{D}]$. Zero objects and biproducts in $[\mathsf{C}, \mathsf{D}]$ are inherited "pointwise" from zero objects and biproducts in D. For example, given a zero object z in D, a zero object Z in $[\mathsf{C}, \mathsf{D}]$ is defined by setting $Z(x) = z$ for all objects $x \in \mathsf{C}$ (and to any morphism, Z assigns the zero morphism $z \to z$.) Similarly, given additive functors $F, G : \mathsf{C} \to \mathsf{D}$, define their biproduct by choosing, for every object $x \in \mathsf{C}$, a biproduct $(F \oplus G)(x) := F(x) \oplus G(x)$ in D, and for every morphism $f : x \to x'$ in C, set $(F \oplus G)(f) := F(f) \oplus G(f)$.

If *both* C and D are additive categories, then the category $[\mathsf{C}, \mathsf{D}]_{\text{add}}$ of additive functors from C to D is also again additive.

As a final remark, we will also make use of the fact that if C is additive, then so is the category $\text{End}(\mathsf{C})$ (recall that objects in this category are pairs (x, f), where $x \in \text{ob}(\mathsf{C})$ and $f \in \text{End}(x)$). To see this, it is sufficient to note that $\text{End}(\mathsf{C})$ corresponds to the functor category $[\mathbb{N}, \mathsf{C}]$, where $\mathbb{N}$ denotes here the category with one object and with hom-set given by the natural numbers (with addition as composition). If we replace $\mathbb{N}$ with $\mathbb{Z}$, then $[\mathbb{Z}, \mathsf{C}]$ corresponds to $\text{Aut}(\mathsf{C})$.

1.5.3. Additive categories and opposites. Since opposites play an important role throughout this thesis, we discuss here some details related to taking opposites in the additive setting.

First of all, the opposite category of an additive category C is, in a natural manner, again additive. To see this, first note that Ab is symmetric monoidal in an obvious way, and so the opposite of C is also Ab-enriched (as discussed in Section 1.5.1). In particular, given $f, g \in \text{Hom}_C(x, y)$, we have, *by definition*,

$$f^\circ + g^\circ = (f + g)^\circ \tag{62}$$

in $\text{Hom}_{C^\circ}(y^\circ, x^\circ)$. Then, note that the definitions of a zero object and of a binary biproduct are self-dual. Indeed, $z \in \mathsf{C}$ is initial/terminal if and only if $z^\circ \in \mathsf{C}^\circ$ is terminal/initial, and given a binary biproduct

$$x \underset{i_x}{\overset{p_x}{\leftrightarrows}} x \oplus y \underset{i_y}{\overset{p_y}{\rightleftarrows}} y$$

in C, we have an associated biproduct

$$x^\circ \underset{(p_x)^\circ}{\overset{(i_x)^\circ}{\leftrightarrows}} (x \oplus y)^\circ \underset{(p_y)^\circ}{\overset{(i_y)^\circ}{\rightleftarrows}} y^\circ$$

in C°, since

$$(i_x)^\circ (p_x)^\circ = (p_x i_x)^\circ = (1_x)^\circ = 1_{x^\circ}, \quad (i_y)^\circ (p_y)^\circ = (p_y i_y)^\circ = (1_y)^\circ = 1_{y^\circ}$$

and

$$(p_x)^\circ(i_x)^\circ + (p_y)^\circ(i_y)^\circ = (i_x p_x)^\circ + (i_y p_y)^\circ = (i_x p_x + i_y p_y)^\circ = 1^\circ_{x\oplus y} = 1_{(x\oplus y)^\circ}.$$

For simplicity, we will always assume $x^\circ \oplus y^\circ = (x\oplus y)^\circ$, even though, technically, these two biproducts need only be canonically isomorphic. So we have $(i_x)^\circ = p_{x^\circ}$ and $(p_x)^\circ = i_{x^\circ}$, and similarly for y and y°.

Now we turn to functors. If $F : \mathsf{C} \to \mathsf{D}$ is additive, then $F^\circ : \mathsf{C}^\circ \to \mathsf{D}^\circ$ is also additive (this follows quite directly from the definition of F°). In particular F° comes endowed with coherence isomorphisms

$$\varphi^{F^\circ}_{x^\circ,y^\circ} : F^\circ(x^\circ \oplus y^\circ) \longrightarrow F^\circ(x^\circ) \oplus F^\circ(y^\circ). \tag{63}$$

LEMMA 1.5.12. *Let $F : \mathsf{C} \to \mathsf{D}$ be an additive functor. Then the coherence isomorphisms (63) for F° are related to the ones for F via*

$$(\varphi^{F^\circ}_{x^\circ,y^\circ})^\circ = (\varphi^F_{x,y})^{-1}. \tag{64}$$

PROOF. The morphism $\varphi^{F^\circ}_{x^\circ,y^\circ}$ is the universal morphism for a certain diagram, of the type (57), for which the corresponding opposite diagram (which is of type (58)) is precisely the one for which $(\varphi^F_{x,y})^{-1}$ is the corresponding universal morphism. Therefore, in the latter diagram, by the uniqueness of such universal morphisms, $(\varphi^{F^\circ}_{x^\circ,y^\circ})^\circ$ must coincide with $(\varphi^F_{x,y})^{-1}$. □

1.5.4. Decompositions, Subobjects, Idempotents. We fix an additive category C and let $x \in \mathrm{ob}(\mathsf{C})$.

By a **decomposition** of x we mean an isomorphism $x \simeq x_1 \oplus \cdots \oplus x_m$ for some finite number m of objects, which we call the **summands** of the decomposition. We say that a decomposition is non-trivial if none of the summands are the zero object. In the following we will focus mainly on binary decompositions, i.e. those with two summands, and how these relate to idempotents in the endomorphism ring $\mathrm{End}(x)$. The object x is called **indecomposable** if it admits no non-trivial decomposition $x \simeq u \oplus w$. In other words, for indecomposable x, if $x \simeq u \oplus w$, then necessarily u or w is a zero object.

Note that any decomposition $x \simeq u \oplus w$ induces an associated "representation" (i.e. instantiation) of x as a biproduct

$$u \underset{i_u}{\overset{p_u}{\leftrightarrows}} x \underset{i_w}{\overset{p_w}{\rightleftarrows}} w, \tag{65}$$

and vice versa. Given another representation

$$u' \underset{i'_u}{\overset{p'_u}{\leftrightarrows}} x \underset{i'_w}{\overset{p'_w}{\rightleftarrows}} w', \tag{66}$$

we say that the two representations (and the associated decompositions) are **equivalent** if there exists isomorphisms $\varphi : u \to u'$ and $\psi : w \to w'$ such that

$$\begin{array}{ccccc} u & \underset{i_u}{\overset{p_u}{\leftrightarrows}} & x & \underset{i_w}{\overset{p_w}{\rightleftarrows}} & w \\ \varphi\downarrow & & \| & & \downarrow\psi \\ u' & \underset{i'_u}{\overset{p'_u}{\leftrightarrows}} & x & \underset{i'_w}{\overset{p'_w}{\rightleftarrows}} & w' \end{array} \tag{67}$$

commutes. To say that two decomposition $x \simeq u \oplus w$ and $x \simeq u' \oplus w'$ are equivalent, we write $u \oplus w = u' \oplus w'$.

LEMMA 1.5.13. *Let $x \simeq u \oplus w$, and suppose $u \overset{\varphi}{\simeq} u'$ and $w \overset{\psi}{\simeq} w'$ in C for some objects $u', w' \in \mathsf{C}$. Then $x \simeq u' \oplus w'$ in an obvious manner, and this decomposition is equivalent to $x \simeq u \oplus w$.*

PROOF. Given (65), it is straightforward to check that $p_{u'} := \varphi p_u$, $i_{u'} := i_u \varphi^{-1}$, $p_{w'} := \psi p_w$, $i_{w'} := i_w \psi^{-1}$ exhibits x as the biproduct $u' \oplus w'$. □

LEMMA 1.5.14. *Let $x \simeq u \oplus w$ and $x \simeq u' \oplus w'$ be decompositions. They are equivalent if and only if $u \overset{\varphi}{\simeq} u'$ and $w \overset{\psi}{\simeq} w'$ as subobjects of x.*

PROOF. If the decompositions are equivalent, then from the definition is follows that the respective summands are isomorphic as subobjects. For the converse statement, we know that because the decomposition $x \simeq u' \oplus w'$ is already given, it comes with the maps $p_{u'}$, $i_{u'}$, $p_{w'}$, $i_{w'}$ for the biproduct already specified, i.e. we are not free to choose them as we did in the proof of the previous lemma. However, the assumption that the two decompositions are equivalent guarantees that the equations which were defining equations there are equations which are true here. □

REMARK 1.5.15. The pedantry in the two above lemmas is due to the fact that two subobjects $u', w' \leq x$ may be isomorphic to u and w, respectively, in C, but not isomorphic *as subobjects* of x.

A morphism $e \in \mathrm{End}(x)$ in the endomorphism ring of an object x is an **idempotent** if $e \circ e = e$. Note that the ring $\mathrm{End}(x)$ always has the idempotents 1_x and 0_x (and these coincide if and only if x is a zero object). We call an idempotent **non-trivial** if it is neither 1_x nor 0_x. If an idempotent e is invertible, it is necessarily 1_x, since $e \circ e^{-1} = 1_x$ implies $e = e \circ e \circ e^{-1} = e \circ e^{-1} = 1_x$.

If e is an idempotent, then $1_x - e$ is one too; we call it the **conjugate** idempotent of e (note that also e is the conjugate of $1_x - e$). In particular, 1_x and 0_x are a pair of conjugate idempotents. For any idempotent, $e(1 - e) = e - e \circ e = 0$.

Let $\mathrm{Idem}(x)$ denote the set of idempotents in $\mathrm{End}(x)$. It may be endowed with a partial order by defining

$$e \leq f \overset{def}{\Leftrightarrow} ef = e = fe. \tag{68}$$

The operation $e \mapsto 1 - e$ defines a strict duality involution on the poset $\mathrm{Idem}(x)$. Indeed, $e = 1 - (1 - e)$, and if $e \leq f$, then

$$(1 - f)(1 - e) = 1 - f - e + fe = 1 - f = (1 - e)(1 - f),$$

i.e. $(1 - f) \leq (1 - e)$.

LEMMA 1.5.16. *To any decompostion $x \simeq u \oplus w$ there is an associated pair of conjugate idempotents e_u and e_w of x. If the decomposition is non-trivial, then so are the idempotents.*

PROOF. If φ is an isomorphism $x \to u \oplus w$, let $e_u := \varphi^{-1} i_u p_u \varphi$ and $e_w := \varphi^{-1} i_w p_w \varphi$. These are idempotents in $\mathrm{End}(x)$ since $i_u p_u$ and $i_w p_w$ are idempotents in $\mathrm{End}(u \oplus w)$, and they are conjugate (all this follows from the properties of the biproduct). Furthermore, if neither u nor w are the zero object, then $i_u p_u$ and $i_w p_w$ (and hence also e_u and e_w) are non-trivial: if $i_u p_u = 0_{u \oplus w}$ were the case, say, then $0_u = p_u i_u p_u = 1_u p_u = p_u$, and so $1_u = p_u i_y u = 0_u$, which implies that u is a zero object, a contradiction. And if

$i_u p_u = 1_{u \oplus w}$, then $i_w p_w = 1_{u \oplus w} - i_u p_u = 0_{u \oplus w}$, and the same reasoning as before implies that w is a zero object, again a contradiction. □

COROLLARY 1.5.17. *If End(x) is a local ring, then x is indecomposable.*

PROOF. Suppose x were to have an non-trivial decomposition. Then by the above Lemma, there would exist a non-trivial idempotent $e \in \mathrm{End}(x)$. We may write $1_x = e + (1_x - e)$, which, since End(x) is local, implies that either e or $1 - e$ is invertible. But this implies that either $e = 1_x$ or $1_x - e_x = 1_x$ (so $e_x = 0_x$). Either way, this contradicts the non-triviality of e. □

LEMMA 1.5.18. *Let $x \simeq u \oplus w$ and $x \simeq u' \oplus w'$ be decompositions with associated pairs of conjugate idempotents (e_u, e_w) and $(e_{u'}, e_{w'})$. If the decompositions are equivalent, then $e_u = e_{u'}$ and $e_w = e_{w'}$.*

PROOF. If the decompositions are equivalent, there are isomorphisms $\varphi : u \to u'$ and $\psi : w \to w'$ such as in (67). In particular then

$$e_u = i_u p_u = i_{u'} \varphi p_u = i_{u'} p_{u'} = e_{u'}.$$

And, of course, $e_u = e_{u'}$ implies $e_w = 1 - e_u = 1 - e_{u'} = e_{w'}$. □

An idempotent e **splits** if there exists $y \in \mathrm{ob}(\mathsf{C})$ and morphisms

$$x \xrightarrow{p_y} y \xrightarrow{i_y} x \tag{69}$$

such that $p_y i_y = 1_y$ and $i_y p_y = e$. In this case, such i_y and p_y are monic and epic, respectively, and y is "the" image of e.

We say that "all idempotents of C split" if for every object x, every idempotent $e \in \mathrm{End}(x)$ splits.

LEMMA 1.5.19. *Suppose $e : x \xrightarrow{p} u \xrightarrow{i} x$ and $e' : x \xrightarrow{p'} u' \xrightarrow{i'} x$ are split idempotents in End(x). If $e \leq e'$, then $u \leq u'$ as subobjects of x.*

PROOF. If $e \leq e'$, then in particular $i'p'ip = e'e = e = ip$. Using the right-cancellation property of the epic p, we find that $i'p'i = i$, which shows that $p'i : u \to u'$ is a morphism of subobjects. □

LEMMA 1.5.20. *Let $e \in$ End(x) be an idempotent. Suppose that both e and $1_x - e$ split, with associated objects and maps*

$$x \xrightarrow{p_y} y \xrightarrow{i_y} x \quad \textit{and} \quad x \xrightarrow{p_z} z \xrightarrow{i_z} x. \tag{70}$$

Then $x \simeq y \oplus z$.

PROOF. We have $p_y i_y + p_z i_z = e + (1_x - e) = 1_x$. And from the definition of "splitting", it follows that the other equations in the definition of the biproduct $x \oplus y$ are also satisfied by the maps i_y, p_y, i_z and p_z. □

REMARK 1.5.21. Suppose that, for the idempotents e and $1_x - e$ of the previous lemma, there exists another pair of splittings

$$x \xrightarrow{p_{y'}} y' \xrightarrow{i_{y'}} x \quad \text{and} \quad x \xrightarrow{p_{z'}} z' \xrightarrow{i_{z'}} x \tag{71}$$

which are, respectively, isomorphic to the splittings (70), in the sense of factorizations. Then it is easy to see that the corresponding decomposition $x \simeq y' \oplus z'$ will be equivalent to the one $x \simeq y \oplus z$.

LEMMA 1.5.22. *Suppose e is an idempotent of x which splits as $x \xrightarrow{p_y} y \xrightarrow{i_y} x$. If p_y has a kernel $k \xrightarrow{k} x$, then the conjugate idempotent $1 - e$ has a splitting $x \xrightarrow{t} k \xrightarrow{k} x$. In particular, we have $x \simeq im(e) \oplus ker(e)$.*

PROOF. Observe that

$$x \xrightarrow{1-e} x \xrightarrow{p_y} y$$

is the zero arrow, since $p_y(1-e) = p_y(1-i_y p_y) = p_y - (p_y i_y)p_y = 0$. Therefore there exists a unique map $t : x \longrightarrow x$ such that

$$\begin{array}{ccccc} x & \xrightarrow{1-e} & x & \xrightarrow{p_y} & y \\ {\scriptstyle t}\downarrow & \nearrow_{k} & & & \\ k & & & & \end{array}$$

commutes. In particular, $kt = 1 - e$. To see that $tk = 1_x$, note that

$$ktk = (1 - i_y p_y)k = k - i_y(p_y k) = k$$

and use the left-cancellation property given by the fact that k is a monomorphism. □

DEFINITION 1.5.23. *An additive category C is **idempotent complete** if for every idempotent $e \in End(x)$ there exists a decomposition*

$$x \simeq im(e) \oplus ker(e)$$

such that

$$e \simeq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

LEMMA 1.5.24. *An additive category is idempotent complete if, and only if, every idempotent has a kernel.*

PROOF. See [**Bue10**], Section 6. □

1.5.5. Abelian categories. The prototypical example for the following definition is the category Ab of abelian groups. An **abelian category** is an additive category C for which

(1) every morphism has a kernel and a cokernel,
(2) every monomorphism is a kernel, and every epimorphism is a cokernel.

One consequence of this definition is that any abelian category C has all finite limits and colimits, since C has all finite (co)products and all (co)equalizers: (co)products because C is additive, and (co)equalizers because, for given parallel morphisms f, g, their (co)equalizer is the (co)kernel of $f - g$.

Other basic, useful facts are that a morphism in abelian C which is both monic and epi is necessarily an isomorphism, and that any morphism f has a factorization

$$f = me \tag{72}$$

with m monic and e epi, and this factorization is unique in an appropriate sense.

Besides the category Ab, another important example of an abelian category is the category $\mathsf{R\text{-}Mod}$ of left R-modules for any ring R. Similarly, also the category $\mathsf{Mod\text{-}R}$ of right R-modules is abelian. Kernels and cokernels correspond here to the usual notions.

A simple (albeit slightly artificial) example of an additive category which is *not* abelian is the full subcategory of $\mathsf{vect}_{\mathbf{k}}$ (finite dimensional vector spaces) whose objects are *even*-dimensional. In this case, the kernel of a linear map which has odd rank will be odd-dimensional, and hence not a kernel in this category. Another category which is additive but not abelian is the category of (finite-dimensional) representations of a (fixed) finite poset P; this category will play an important role in Chapter 7.

Part 1

Categories with duality

In this part of the thesis, we introduce the basic theory of duality involutions and their fixed points, including various examples and constructions. Much of the material is taken or inspired from the references [**QSS79**], [**Knu91**], [**Shi12**], [**Jac12**], [**FH16**], although we are unaware of references which develop a general theory of duality involutions. The list of relevant examples and constructions is long; for lack of time we have left many unmentioned, and surely there are many more which we are as yet unaware of.

CHAPTER 2

Duality involutions

In this chapter, we follow the convention that if C is a category, then C° denotes its opposite. We will sometimes write x° or f° to indicate the object or morphism, respectively, in C° corresponding to an object x or morphism f in C. Similarly, if $x \in \mathsf{C}^\circ$, then $x^\circ \in \mathsf{C}$, etc..

2.1. Basic notions

DEFINITION 2.1.1. *A **duality involution** on a category C is a pair (δ, η), where $\delta : \mathsf{C} \to \mathsf{C}^\circ$ is a functor and η is a natural transformation $1_{\mathsf{C}} \overset{\eta}{\Rightarrow} \delta^\circ\delta$ such that, for all $x \in ob(\mathsf{C})$, the following commutes:*

(73)
$$\begin{array}{ccc} \delta^\circ x^\circ & \xrightarrow{\eta_{\delta^\circ x^\circ}} & \delta^\circ\delta\delta^\circ x^\circ \\ & {\scriptstyle 1_{\delta^\circ x^\circ}} \searrow & \downarrow {\scriptstyle \delta^\circ(\eta_x)^\circ} \\ & & \delta^\circ x^\circ \end{array}$$

*If η is a natural isomorphism, then we say that (δ, η) is a **strong** duality involution; if η is an equality, (δ, η) is called **strict**.*

*The data $(\mathsf{C}, \delta, \eta)$ of a category equipped with a duality involution will be called a **category with duality**.*

REMARK 2.1.2.

(1) To parse (73), note that $(\eta_x)^\circ : (\delta^\circ\delta x)^\circ = \delta\delta^\circ x^\circ \longrightarrow x^\circ$.
(2) The diagrams (73) say, in other words, that we have the following commutative diagram of natural transformations

(74)
$$\begin{array}{ccc} \delta^\circ & \xrightarrow{\eta\star 1_{\delta^\circ}} & \delta^\circ\delta\delta^\circ \\ & {\scriptstyle 1_{\delta^\circ}} \searrow & \downarrow {\scriptstyle 1_{\delta^\circ}\star\eta^\circ} \\ & & \delta^\circ \end{array}$$

The definition of duality involution implies that we have an adjunction

$$\mathsf{C} \underset{\delta^\circ}{\overset{\delta}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{C}^\circ,$$

where η is the unit of the adjunction, and η° the counit. Indeed, (74) is one of the "triangle identities" (it is the one (53)), and the other triangle identity is the opposite of the diagram (74), i.e.

(75)
$$\begin{array}{ccc} \delta & \xleftarrow{\eta^\circ \star 1_\delta} & \delta\delta^\circ\delta \\ & {\scriptstyle 1_\delta}\nwarrow & \uparrow {\scriptstyle 1_\delta \star \eta} \\ & & \delta \end{array}$$

◇

We note that $\delta \dashv \delta^\circ$ does not imply in general that $\delta^\circ \dashv \delta$ (see [**ML98**], p. 88).

Example 2.1.3. If C is a groupoid, i.e. a category in which every morphism is invertible, then C may be equipped with the duality involution defined by setting $\delta x = x^\circ$ and $\delta f = (f^{-1})^\circ$, letting η be defined in components by $\eta_x := 1_x$.

Example 2.1.4. Consider $\mathsf{C} = \mathsf{Vect}_{\mathbf{k}}$, the category of vector spaces over a fixed ground field $\mathbf{k}$. A duality involution is defined via the usual duality given by

$$\delta V := V^* = \mathrm{Hom}_{\mathsf{C}}(V, \mathbf{k})$$

on objects, and

$$\delta f := f^* \in \mathrm{Hom}_{\mathsf{C}^\circ}(V^*, W^*) = \mathrm{Hom}_{\mathsf{C}}(W^*, V^*)$$

for a morphism $f \in \mathrm{Hom}_{\mathsf{C}}(V, W)$, where f^* is the usual adjoint map

$$W^* \longrightarrow V^*, \xi \longmapsto \xi \circ f.$$

For a duality involution we also need to specify a unit η; component-wise we take this to be

(76)
$$\eta_V : V \longrightarrow \delta^\circ\delta V = V^{**},\ v \longmapsto \iota_v : \xi \mapsto \xi(v) \qquad V \in \mathsf{vect}_{\mathbf{k}}.$$

To see that the condition (73) is satisfied, we check that the diagram

(77)
$$\begin{array}{ccc} (V^\circ)^* & \xrightarrow{\eta_{(V^\circ)^*}} & (V^\circ)^{***} \\ & {\scriptstyle 1_{(V^\circ)^*}}\searrow & \downarrow {\scriptstyle (\eta_V)^*} \\ & & (V^\circ)^* \end{array}$$

commutes for every vector space V. The top path through the diagram is the map

(78)
$$\xi \longmapsto \iota_\xi \circ \eta_V \qquad \xi \in (V^\circ)^*.$$

And

$$(\iota_\xi \circ \eta_V)(x) = \iota_\xi(\eta_V(x)) = \iota_\xi(\iota_x) = \iota_x(\xi) = \xi(x) \qquad x \in V^\circ,$$

so (78) is indeed the identity on $(V^\circ)^*$, as desired. △

We will call the duality involution of Example 2.1.4 above the "standard duality involution" on vector spaces.

Remark 2.1.5. A variant of the previous example which will be of importance to us later is the following. Let everything be the same as above, except modify the definition of η such that its components are

(79)
$$\eta_V : V \longrightarrow V^{**},\ v \longmapsto -\iota_v : \xi \mapsto -\xi(v) \qquad V \in \mathsf{Vect}_{\mathbf{k}}.$$

In other words, with respect to the previous example, we have multiplied the linear maps which were the components of the unit by -1. This again defines a duality involution (it is clear from the calculation in the previous example that (73) is satisfied). $\diamond$

Since both the duality involution of Example 2.1.4 and the variant with (79) as unit will play an important role in this thesis, we introduce a notation to speak of both variants at the same time.

Definition 2.1.6. *Let $\varepsilon \in \{-,+\}$. The notation $\mathsf{Vect}^{\varepsilon}_{\mathbf{k}}$ will denote the category with duality $(\mathsf{Vect}_{\mathbf{k}}, \delta, \eta)$ where δ is as above and η is either (76) or (79), depending on whether $\varepsilon = +$ or $\varepsilon = -$, respectively.*

Now we return to the general theory, and discuss morphisms between categories with duality.

Definition 2.1.7. *A morphism, or* ***equivariant functor****, $(\mathsf{C}, \delta, \eta) \longrightarrow (\mathsf{D}, \delta', \eta')$ between categories with duality involution is a pair (F, ψ), where $F : \mathsf{C} \to \mathsf{D}$ is a functor, and ψ is a natural transformation*

$$\begin{array}{ccc} \mathsf{C} & \xrightarrow{\;F\;} & \mathsf{D} \\ {\scriptstyle\delta}\big\downarrow & \overset{\psi}{\Swarrow} & \big\downarrow{\scriptstyle\delta'} \\ \mathsf{C}^{\circ} & \xrightarrow[F^{\circ}]{} & \mathsf{D}^{\circ} \end{array}$$

such that the following equation holds

$$(\delta'^{\circ}\psi) \circ \eta' F = (\psi^{\circ}\, \delta) \circ F\eta. \tag{80}$$

An equivariant functor (F, ψ) is ***strong*** *if ψ is a natural isomorphism, and* ***strict*** *if ψ is an equality.*

We will sometimes refer to a morphism (F, ψ) simply as F.

Remark 2.1.8. Note that the components of ψ are, by definition, morphisms

$$\psi_x : \delta' F x \longrightarrow F^{\circ}\delta x \qquad x \in \mathsf{C}$$

in the *opposite* of D. The corresponding morphisms in D are

$$(\psi_x)^{\circ} : (F^{\circ}\delta x)^{\circ} \longrightarrow (\delta' F x)^{\circ} \qquad x \in \mathsf{C}$$

or, equivalently,

$$\psi^{\circ}_{x^{\circ}} : F\delta^{\circ} x^{\circ} \longrightarrow \delta'^{\circ} F^{\circ} x^{\circ} \qquad x \in \mathsf{C}.$$

Condition (80) says that the diagram

$$\begin{array}{ccc} Fx & \xrightarrow{\;F\eta_x\;} & F\delta^{\circ}\delta x \\ {\scriptstyle \eta'_{Fx}}\big\downarrow & & \big\downarrow{\scriptstyle \psi^{\circ}_{\delta x}} \\ \delta'^{\circ}\delta' F x & \xrightarrow[\delta'^{\circ}\psi_x]{} & \delta'^{\circ} F^{\circ}\delta x \end{array} \tag{81}$$

commutes for all $x \in \mathsf{C}$.

Example 2.1.9.

(1) Let $(\mathsf{C}, \delta, \eta)$ be a category with duality involution. The identity functor on C is equivariant when equipped with the identity natural transformation $\psi = \mathrm{id}_{\delta} : \delta \Rightarrow \delta$.

(2) Consider the category $\mathsf{Vect}_{\mathbf{k}}$ with the standard duality involution and consider the functor $F : \mathsf{Vect}_{\mathbf{k}} \to \mathsf{Vect}_{\mathbf{k}}$ which acts on objects by $V \longmapsto V \times V$ and on morphisms by $f \longmapsto f \times f$. To define $\psi : \delta F \Rightarrow F^\circ \delta$ we can equivalently define $\psi^\circ : F\delta^\circ \Rightarrow \delta^\circ F^\circ$ (this being more convenient, since the components of ψ° are morphisms in $\mathsf{Vect}_{\mathbf{k}}$). We set

$$\psi_V^\circ : V^* \times V^* \longrightarrow (V \times V)^*, (\xi, \zeta) \longmapsto [(v, w) \mapsto \xi(v) + \zeta(w)],$$

thinking of V here as living in $\mathsf{Vect}_{\mathbf{k}}^\circ$. We claim that (F, ψ) is equivariant. To show this, we check (81); i.e. that the square

$$\begin{array}{ccc} V \times V & \xrightarrow{\eta_V \times \eta_V} & V^{**} \times V^{**} \\ {\scriptstyle \eta_{V\times V}}\downarrow & & \downarrow{\scriptstyle \psi^\circ_{V^*}} \\ (V \times V)^{**} & \xrightarrow[(\psi_V)^*]{} & (V^* \times V^*)^* \end{array} \tag{82}$$

commutes. Let $(v, w) \in V \times V$. The upper path through the diagram maps (v, w) to the functional in $(V^* \times V^*)^*$ given by

$$(\xi, \zeta) \longmapsto \psi^\circ_{V^*}(\eta_V v, \eta_V w)(\xi, \zeta) = (\eta_V v)(\xi) + (\eta_V w)(\zeta) = \xi(v) + \zeta(w).$$

The lower path, on the other hand, maps (v, w) to the functional

$$(\xi, \zeta) \longmapsto \psi^\circ_V(\xi, \zeta)(v, w) = \xi(v) + \zeta(w).$$

△

Definition 2.1.10. *Let (F, ψ_F) and (G, ψ_G) be morphisms*

$$(\mathcal{C}, \delta, \eta) \longrightarrow (\mathcal{D}, \delta', \eta')$$

between categories with duality. A ***natural transformation*** *from (F, ψ_F) to (G, ψ_G) is a natural transformation $\alpha : F \Rightarrow G$ such that the following diagram*

$$\begin{array}{ccc} \delta' \circ F & \xrightarrow{\psi_F} & F^\circ \circ \delta \\ {\scriptstyle \delta'\alpha}\downarrow & & \uparrow{\scriptstyle \alpha^\circ\delta} \\ \delta' \circ G & \xrightarrow[\psi_G]{} & G^\circ \circ \delta \end{array}$$

commutes.

Morphisms between categories with duality and their natural transformations can be appropriately composed as follows.

Definition 2.1.11. *Given morphisms*

$$(\mathcal{C}, \delta, \eta) \overset{(F,\psi_F)}{\longrightarrow} (\mathcal{D}, \delta', \eta') \overset{(G,\psi_G)}{\longrightarrow} (\mathcal{E}, \delta'', \eta''),$$

we define their composition to be the pair

$$(G \circ F, G^\circ \psi_F \circ \psi_G F). \tag{83}$$

Lemma 2.1.12. *(83) is a morphism*

$$(\mathcal{C}, \delta, \eta) \longrightarrow (\mathcal{E}, \delta'', \eta'').$$

Proof. We have to check that $G^\circ \psi_F \circ \psi_G F$ preserves the unit. It is a consequence of the interchange law of natural transformations. We leave the straightforward yet tedious proof to the reader. □

COROLLARY 2.1.13. *Suppose we are given composable equivariant functors. If they are strongly equivariant, then so is their composite; if they are strictly equivariant, then so is their composite.*

REMARK 2.1.14. The composition of morphisms is strictly associative: indeed, for (F, ψ_F), (G, ψ_G) and (H, ψ_H) the following equation holds

$$H^\circ(G^\circ\psi_F \circ \psi_G F) \circ \psi_H(G \circ F) = ((H \circ G)^\circ\psi_F \circ (H^\circ\psi_G) \circ (\psi_H G)) \circ F \tag{84}$$

We now turn to natural transformations of morphisms between categories with duality. These may be composed vertically and horizontally.

LEMMA 2.1.15. *Let (F, ψ_F), (G, ψ_G) and (H, ψ_H) be morphisms*

$$(\mathsf{C}, \delta, \eta) \longrightarrow (\mathsf{D}, \delta', \eta'),$$

and let $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ be natural trasformations. The composition $\beta \circ \alpha$ is a natural transformation between (F, ψ_F) and (H, ψ_H).

PROOF. We have to show that the (vertical) composition $\beta \circ \alpha$ satisfies the relevant commutative diagram. Consider then the following diagram

$$\begin{array}{ccc}
\delta' \circ F & \xrightarrow{\psi_F} & F^\circ \circ \delta \\
{\scriptstyle \delta'\alpha}\downarrow & & \uparrow{\scriptstyle \alpha^\circ\delta} \\
\delta' \circ G & \xrightarrow{\psi_G} & G^\circ \circ \delta \\
{\scriptstyle \delta'\beta}\downarrow & & \uparrow{\scriptstyle \beta^\circ\delta} \\
\delta' \circ H & \xrightarrow[\psi_H]{} & H^\circ \circ \delta
\end{array}$$

Since the inner sub-diagrams are commutative, so is the outer one. Since $(\delta'\beta) \circ (\delta\alpha) = \delta'(\beta \circ \alpha)$ and $(\alpha^\circ\delta) \circ (\beta^\circ\delta) = (\beta \circ \alpha)^\circ\delta$, the outer diagrams is the desired compatibility diagram. □

LEMMA 2.1.16. *Let (F, ψ_F), (G, ψ_G) be morphisms*

$$(\mathsf{C}, \delta, \eta) \longrightarrow (\mathsf{D}, \delta', \eta'),$$

and let $(F', \psi_{F'})$, $(G', \psi_{G'})$ be morphisms

$$(\mathsf{D}, \delta', \eta') \longrightarrow (\mathcal{E}, \delta'', \eta'').$$

Let $\alpha : F \Rightarrow G$ and $\beta : F' \Rightarrow G'$ be natural transformations. Then the horizontal composition $\beta \star \alpha$ is a natural transformation between the equivariant functors $F' \circ F$ and $G' \circ G$.

PROOF. Recall that the horizontal composition $\beta \star \alpha$ can be rephrased as $(G'\alpha) \circ (\beta F)$. By whiskering on the right with F and on the left with G'°, the compatibility diagrams for α and β induce the following commutative diagrams

$$\begin{array}{ccc}
\delta'' F' F & \xrightarrow{\psi_{F'} F} & F'^\circ \delta' F \\
{\scriptstyle \delta''\beta F}\downarrow & & \uparrow{\scriptstyle \beta^\circ\delta' F} \\
\delta'' G' F & \xrightarrow[\psi_{G'} F]{} & G'^\circ \delta' F
\end{array}
\qquad
\begin{array}{ccc}
G'^\circ \delta' F & \xrightarrow{G'^\circ\psi_F} & G'^\circ F^\circ \delta \\
{\scriptstyle G'^\circ\delta'\alpha}\downarrow & & \uparrow{\scriptstyle G'^\circ\alpha^\circ\delta F} \\
G'^\circ \delta' G & \xrightarrow[G'^\circ\psi_G]{} & G'^\circ G^\circ \delta
\end{array}$$

On the other hand, the interchange law for natural transformations provides the following commutative squares

$$
\begin{CD}
\delta''G'F @>{\psi_{G'F}}>> G'^{\circ}\delta'F \\
@V{\delta''G'\alpha}VV @VV{G'^{\circ}\delta'\alpha}V \\
\delta''G'G @>>{\psi_{G'G}}> G'^{\circ}\delta'G
\end{CD}
\qquad\qquad
\begin{CD}
F'^{\circ}\delta'F @>{F'^{\circ}\psi_F}>> F'^{\circ}F^{\circ}\delta \\
@A{\beta^{\circ}\delta'F}AA @AA{\beta^{\circ}F^{\circ}\delta}A \\
G'^{\circ}\delta'F @>{G'^{\circ}\psi_F}>> G'^{\circ}F^{\circ}\delta
\end{CD}
$$

Combining the diagrams above we obtain the following diagram

$$
\begin{CD}
\delta''F'F @>{\psi_{F'F}}>> F'^{\circ}\delta'F @>{F'^{\circ}\psi_F}>> F'^{\circ}F^{\circ}\delta \\
@V{\delta''\beta F}VV @AA{\beta^{\circ}\delta'F}A @AA{\beta^{\circ}F^{\circ}\delta}A \\
\delta''G'F @>{\psi_{G'F}}>> G'^{\circ}\delta'F @>>{G'^{\circ}\psi_G}> G'^{\circ}G^{\circ}\delta \\
@V{\delta''G'\alpha}VV @V{G'^{\circ}\delta'\alpha}VV @AA{G'^{\circ}\alpha^{\circ}\delta F}A \\
\delta''G'G @>>{\psi_{G'G}}> G'^{\circ}\delta'G @>>{G'^{\circ}\psi_G}> G'^{\circ}G^{\circ}\delta
\end{CD}
$$

Notice that the upper and lower external horizontal legs of the outer diagram corresponds to the equivariance data for $F'F$ and $G'G$, respectively. On the other hand, the left and right vertical legs correspond to $\delta''(\beta \star \alpha)$ and $(\beta \star \alpha)^{\circ}\delta$. The outer diagram corresponds then to the compability condition for the natural transformation $\beta \star \alpha$. We it leave to the reader to check that the outer diagram is commutative given the commutativity of the internal ones. □

The above lemma guarantee the following

PROPOSITION 2.1.17. *Categories with duality, with their morphisms and natural transformations, form a strict bicategory. We denote it by* dCat.

PROOF. We are left to check that the interchange law for 2-morphisms holds; this is a direct consequence of the interchange law for natural transformations between functors. □

2.2. Examples

2.2.1. Posets. Recall that a **poset** is a set P equipped with a binary relation "$\leq$" which is reflexive, transitive, and anti-symmetric, the latter meaning that if $x \leq y$ and $y \leq x$, then $x = y$. If the anti-symmetry axiom is dropped, we have a **pre-order**. The definition of preordered set is itself subsumed by the notion of a **thin category**: a category where between any two objects there is is at most one morphism. A preordered set is a thin category if we think of its elements as the objects, and we interpret the symbol $x \leq y$ to mean that there exists a (unique) morphism $x \to y$. The existence of identity morphisms encodes the reflexivity axiom and the composition law encodes transitivity axiom. We call a thin category **posetal** if, additionally, isomorphic objects are always equal.

Let A and B be posetal categories. An adjunction between A and B° is known as an **antitone Galois connection**. Duality involutions on posetal categories are thus special kinds of antitone Galois connections. Note that if C is a poset, an endomap δ of its underlying set is a duality involution if and only if

- $x \leq y \;\Rightarrow\; \delta(y) \leq \delta(x)$, and
- $x \leq \delta\delta(x)$.

Given a duality involution δ on a partially ordered set C, the functor $T = \delta^{\circ}\delta$ corresponds to what is called a **closure operator**: an endomap of the underlying set which satisfies

- $x \leq T(x) \quad \forall x \in \mathsf{C}$;
- $x \leq y \;\Rightarrow\; T(x) \leq T(y)$;
- $T(T(x)) = T(x)$.

EXAMPLE 2.2.1. Fix a set X and consider its powerset $P := \mathcal{P}(X)$, ordered by inclusion. The operation on P defined by

$$A \longmapsto A^c := \{x \in X \mid x \notin A\}$$

defines a (strict) duality involution.

If X is equipped with a topology, then we also have the operations on P of taking the interior, $A \mapsto A^\circ$ and taking the closure, $A \mapsto \overline{A}$ of subsets of X. These operations are compatible with complementation via

$$(A^\circ)^c = \overline{(A^c)}.$$

The operation $\delta(A) := (A^\circ)^c$ then defines a duality involution on P. The associated closure operator $T = \delta^\circ \delta$ acts by

$$T(A) = (((A^\circ)^c)^\circ)^c = \overline{(((A^\circ)^c)^c)} = \overline{(A^\circ)} = \overline{A},$$

i.e. it coincides with the operation of taking the closure.

The following example will play an important role in Part 2 of the thesis.

EXAMPLE 2.2.2. Let V be a finite-dimensional vector space over $\mathbf{k}$ and let $B : V \times V \to \mathbf{k}$ be a blinear form which is either symmetric or skew-symmetric. Then for any subspace $U \subseteq V$ its **orthogonal** is the following subspace of V:

$$U^\perp := \{v \in V \mid B(v,u) = 0 \;\forall u \in U\}. \tag{85}$$

This defines a duality involution

$$U \longmapsto U^\perp$$

on the poset of subspaces of V (where the order-relation is inclusion): if $U \subseteq W$, then $W^\perp \subseteq U^\perp$, and $U \subseteq (U^\perp)^\perp$.

2.2.2. Groups. Let $(G, \cdot, 1)$ be a group. We may view G as a category G having a single object (call it $*$, say), the morphisms of which correspond to the elements of G, and with composition in G given by the group operation in G. If H is another group, then functors

$$F : \mathsf{G} \longrightarrow \mathsf{H}$$

correspond to group homomorphisms $G \longrightarrow H$. A natural transformation $\alpha : F \Rightarrow F'$ between functors $\mathsf{G} \longrightarrow \mathsf{H}$ is the data of a single group element $a := \alpha_* \in H$ such that, for every morphism g in G,

$$\begin{array}{ccc} F(*) & \xrightarrow{F(g)} & F(*) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\ F'(*) & \xrightarrow[F'(g)]{} & F'(*) \end{array}$$

commutes. In other words, for every element $g \in G$,

$$F'(g) = a^{-1} F(g) a,$$

i.e. α corresponds to an inner automorphism C_a of H such that $F = C_a \circ F'$.

Given G, the opposite category corresponds to the **opposite group** G°, i.e. the group $(G, \cdot^\circ, 1)$, where $x \cdot^\circ y := y \cdot x$. The the inversion map of G is a group homomorphism

$$(-)^{-1} : G \longrightarrow G^\circ;$$

this corresponds to a strict duality involution

$$\delta : \mathsf{G} \longrightarrow \mathsf{G}^\circ.$$

Beside the option of taking the identity natural transformation as the unit

$$\eta : 1_{\mathsf{G}} \Longrightarrow \delta^\circ \delta$$

for the duality involution, we may also take η to be any natural transformation whose single component $a \in G$ is in the center of G. This works because C_a then acts as the identity on G, and because for such choice of η, the condition (73) amounts to the commutativity of

(86)
$$\begin{array}{ccc} * & \xrightarrow{a} & * \\ & \searrow^{1_G} & \downarrow a^{-1} \\ & & * \end{array}$$

which is always satisfied.

2.2.3. Vector space categories. Example 2.1.4 and Remark 2.1.5 give prototypical examples of a category with duality involution, and there are many variants of these examples. These variants typically involve a category whose objects are vector spaces equipped with extra structure, and whose morphisms are morphisms of vector spaces which are compatible with the extra structure. As an illustration, we consider the example of representations of a finite group.

EXAMPLE 2.2.3. Let G be a finite group, and let $\mathsf{C} = \mathrm{Rep}(G)$ be the category of (finite-dimensional) group representations of G over the complex number field $\mathbf{k} = \mathbb{C}$. This means that an object of C is a finite-dimensional complex vector space V equipped with a group morphism $\psi : G \longrightarrow \mathrm{Aut}(V)$. Given another representation $\psi' : G \longrightarrow \mathrm{Aut}(W)$, a morphism $\psi \to \psi'$ is a linear map $T : V \to W$ such that $\psi'(g) \circ T = T \circ \psi(g)$ for all $g \in G$.

We may define a duality functor δ as follows: given a representation ψ, define

$$\delta\psi : G \longrightarrow \mathrm{Aut}(V^*), \;\; g \longmapsto (\psi(g)^{-1})^*.$$

Given a morphism T of representations, we set $\delta T = T^*$. To have a duality involution we also need a unit η, the components of which are morphisms $\eta_\psi : \psi \longrightarrow \psi^{**}$. We may take these to be the canonical embedding of V into V^{**} (or the negatives of those maps).

2.2.4. Dagger categories. A dagger category is a category C equipped with a functor $\dagger : \mathsf{C} \to \mathsf{C}^\circ$ such that $\dagger^\circ \dagger = 1_{\mathsf{C}}$ and such that $\dagger$ acts as the identity on objects. Clearly, this is a very special case of a category with duality involution (the unit of the duality involution is the identity natural transformation).

Usually, the action of $\dagger$ on a morphism f is written as $f^\dagger$. The dagger notation stems from following prototypical example of dagger category.

EXAMPLE 2.2.4. Consider the category hilb whose objects are finite-dimensional complex Hilbert spaces $(H, \langle -, - \rangle_H)$, and whose morphisms are linear maps. Given a morphism

$$f : H_1 \longrightarrow H_2,$$

let $f^\dagger$ be the adjoint map of f with respect to the Hermitian inner products in the source and target of f, i.e.

$$f^\dagger = b_1^{-1} f^* b_2$$

where, for $i = 1, 2$, the map $b_i : H_i \to \overline{H}_i^*$ is the one induced by the Hermitian product on H_i.

We have $f^{\dagger\dagger} = f$ since

$$\begin{aligned} f^{\dagger\dagger} &= (b_1^{-1} f^* b_2)^\dagger = b_2^{-1}(b_1^{-1} f^* b_2)^* b_1 = b_2^{-1} b_2^* f^{**} (b_1^{-1})^* b_1 \\ &= (b_2^{-1} b_2^* \iota_2)(\iota_2^{-1} f^{**} \iota_1)(\iota_1^{-1}(b_1^{-1})^* b_1) = \iota_2^{-1} f^{**} \iota_1 = f, \end{aligned}$$

where the $\iota_i : H_i \to H_i^{**}$ are the usual canonical isomorphisms.

The following is another basic example of a dagger category.

Example 2.2.5. Consider the category Rel whose objects are sets and whose morphisms are binary relations, i.e. a morphism $X \to Y$ is a subset $R \subseteq X \times Y$. Given such a morphism R, define $R^\dagger \subseteq X \times X$ to be its **reverse**, i.e.

$$R^\dagger = \{(y, x) \in Y \times X \mid (x, y) \in R\}.$$

Clearly $R^{\dagger\dagger} = R$.

Example 2.2.6. A similar example to the previous one: consider the category of linear relations, where objects are vector spaces over a fixed ground field, and morphisms are linear relations, i.e. a morphism $V \to W$ is a linear subspace $R \subseteq V \oplus W$. Then a dagger operation is defined again by "taking the reverse" as above, i.e.

$$R^\dagger = \{(w, v) \in W \times V \mid (v, w) \in R\}.$$

2.2.5. *-monoids, *-rings, *-algebras. Another large family of examples is given by algebraic structures equipped with an "involution" operation which is "contravariant" with respect to "the multiplication" of the algebraic structure. As an illustrative example, we consider monoids equipped with such an operation.

A ***-monoid** is a monoid M together with a unary operation $(-)^*$ such that $(xy)^* = y^* x^*$ for all $x, y \in M$, and such that $(x^*)^* = x$ for all $x \in M$.

If we think of a monoid as a 1-object category M (so the elements of M are the morphisms in M), then such a $(-)^*$-structure on M corresponds to a (strict) duality involution on M.

Since many algebraic structures, such as rings and algebras, may be understood as monoids internal to an appropriate category, it would be interesting to make precise what it means to have a $*$-monoid internal to a category.

2.2.6. Pontryagin duality. Pontryagin duality is a duality involution on a suitable category of locally compact abelian groups. This restricts to the subcategory of finite abelian groups (finite groups are locally compact when equipped with the discrete topology). For simplicity, we consider the finite group case.

Given a finite abelian group G, its **Pontryagin dual** is the finite abelian group

$$\widehat{G} := \mathrm{Hom}(G, \mathbb{R}/\mathbb{Z}) \tag{87}$$

equipped with the obvious induced point-wise group structure, which is again abelian. Given a morphism $f : G \to H$ of finite abelian groups, its dual is

$$\hat{f} : \widehat{H} \longrightarrow \widehat{G}, \; \chi \longmapsto \chi \circ f. \tag{88}$$

Thus the definition of the duality involution for Pontryagin duality is analogous to the one for vector spaces. For finite abelian groups, the components of the unit of the duality involution are isomorphisms $G \to \widehat{\widehat{G}}$.

2.2.7. Powerset duality. Let $\Omega = \{0, 1\}$ be the set with two elements. Given a set X, its power set $\mathcal{P}(X)$ may be described as the set Ω^X of all functions from X to Ω (a subset $A \subseteq X$ corresponds to the function $X \to \Omega$ which takes the value "1" for elements of A, and "0" else[1]). We take the point of view that Ω^X is "the dual" of X. This may be encoded with the "powerset functor"[2]

$$\delta : \mathsf{Set} \to \mathsf{Set}^\circ \tag{89}$$

which acts on objects by $X \longmapsto \Omega^X$ and acts on morphisms by sending a function $f : X \to Y$ to the function

$$\delta f : \Omega^Y \longrightarrow \Omega^X, \ \chi \longmapsto \chi \circ f. \tag{90}$$

For each set X we also have a canonical function $\eta_X : X \to \Omega^{(\Omega^X)}$ defined by

$$\eta_X : x \longmapsto ev_x : \chi \mapsto \chi(x). \tag{91}$$

It is easily checked that the functions η_X assemble to a natural transformation

$$\eta : 1_{\mathsf{Set}} \Rightarrow \delta^\circ \delta. \tag{92}$$

LEMMA 2.2.7. *With the powerset functor δ, and with η as above, $(\mathsf{Set}, \delta, \eta)$ is a category with (weak) duality.*

PROOF. We need to check that the diagram

$$\begin{array}{ccc} \Omega^X & \xrightarrow{\eta_{\Omega^X}} & \Omega^{(\Omega^{(\Omega^X)})} \\ & \searrow^{=} & \downarrow{\scriptstyle (-)\circ\eta_X} \\ & & \Omega^X \end{array} \tag{93}$$

commutes for each set X. Given $\xi \in \Omega^X$, we have $\eta_{\Omega^X}(\xi) = ev_\xi$, and $ev_\xi \circ \eta_X = \xi$ since for any $x \in X$,

$$ev_\xi \circ \eta_X(x) = ev_\xi(ev_x) = ev_x(\xi) = \xi(x).$$

□

2.2.8. Relations. Various categories whose morphisms are based on some notion of "function" may be enlarged such that morphisms are "relations": by a relation $x \to w$ we mean a subobject $r \subseteq x \times y$, where " $\times$ " is a product. Passing from "functions" to "relations" is typically an "enlargement" in the sense that any "function" $f : x \to y$ may be turned into a relation by considering its graph, i.e. the subobject defined by $x \xrightarrow{\langle 1_x, f \rangle} x \times y$. Rather than trying to make this procedure precise, we give two illustrative examples. In these examples, a duality involution defined on "functions" by "taking the adjoint" may be extended to the setting of relations.

[1] Also recall that for any set A, there is precisely one function $\emptyset \to A$; thus $\Omega^\emptyset$ is has precisely one element.

[2] We are considering the functor known as the contravariant powerset functor. There is also a covariant powerset function, which we do not consider here.

EXAMPLE 2.2.8. We can extend the duality involution defined by the powerset functor on Set (see Section 2.2.7) to the category Rel of relations, which has Set as a subcategory (identitfying functions with their graphs). We extend the duality functor δ to Rel as follows. Given a relation $R \subseteq X \times Y$, define its dual $\delta R \subseteq \Omega^Y \times \Omega^X$ by

$$\delta R = \{(\chi, \zeta) \in \Omega^Y \times \Omega^X \mid \chi(y) = \zeta(x)\ \forall (x, y) \in R\}. \tag{94}$$

Note that if R is the graph of a function f, i.e. $R = \{(x, fx) \mid x \in X\}$, then δR is the graph of δf as defined in (90), since then the condition $\chi(y) = \zeta(x)\ \forall (x, y) \in R$ is equivalent to $\chi(fx) = \zeta(x)$ for all $x \in X$, which means exactly that $\zeta = \chi \circ f = \delta f(\chi)$.

To see that the above really does define a duality involution, observe that Set is a wide subcategory of Rel, i.e. it has the same collection of objects, and observe that the proof of Lemma 2.2.7 does not directly involve the morphisms of the category in question, but only involves the objects and the unit η of the duality functor. Thus to check that the extended duality δ on Rel defines a duality involution, one only needs to verify that η is a natural transformation also in Rel, i.e. one needs to check the naturality squares involving morphisms which are relations, and not just functions. We leave this straightforward verification to the reader. △

EXAMPLE 2.2.9. We can extend the "standard" duality involution on the category of $\mathbf{k}$-vector spaces (Example 2.1.4) to the category of linear relations. For this we can proceed in essentially the same manner as in the the previous example on relations between sets: given a linear relation $R : V \to W$, i.e. a linear subspace $R \subseteq V \oplus W$, we define

$$\delta R = \{(\zeta, \chi) \in W^* \oplus V^* \mid \zeta(w) = \chi(v)\ \forall (v, w) \in R\}, \tag{95}$$

which is again a linear relation. By the same arguments as in the previous example, one sees that this notion of duality generalizes the duality defined on the category of $\mathbf{k}$-vector spaces and linear maps. △

REMARK 2.2.10. Consider the duality involution on linear relations from the previous example. The map

$$\langle -, - \rangle : (V \oplus W) \times (W^* \oplus V^*) \longrightarrow \mathbf{k},\ ((v, w), (\zeta, \chi)) \longmapsto \zeta(w) - \chi(v). \tag{96}$$

is a non-degenerate bilinear pairing, and with respect to this pairing, δR is the annihilator of the linear relation $R \subseteq V \oplus W$, i.e.

$$\delta R = \{(\zeta, \chi) \in W^* \oplus V^* \mid \langle (v, w), (\zeta, \chi) \rangle = 0\ \forall (v, w) \in R\}. \tag{97}$$

This allows to apply standard facts about annihilators – for example, if we restrict ourselves to finite-dimensional vector spaces, it follows that

$$\dim(\delta R) + \dim(R) = \dim(V) + \dim(W) \tag{98}$$

and

$$\dim(\delta^\circ \delta R) = \dim(R). \tag{99}$$

2.3. Constructions

2.3.1. Functor categories. Let $(\mathsf{C}_1, \delta_1, \eta_1)$ and $(\mathsf{C}_2, \delta_2, \eta_2)$ be categories with duality involutions, and let C_1 be essentially small. Then the functor category $[\mathsf{C}_1, \mathsf{C}_2]$ comes naturally endowed with the structure of a duality involution (δ, η). The latter duality involution is strong if and only if the ones on C_1 and C_2 are both strong. In the following we will not make the identification $[\mathsf{C}_1, \mathsf{C}_2]^\circ \simeq [\mathsf{C}_1^\circ, \mathsf{C}_2^\circ]$; rather we work directly with

$[\mathsf{C}_1, \mathsf{C}_2]^\circ$, which, by definition, has the same class of objects as $[\mathsf{C}_1, \mathsf{C}_2]$. We will still use F° to denote the opposite functor of F as an object of $[\mathsf{C}_1^\circ, \mathsf{C}_2^\circ]$; the functor F, viewed as an object of $[\mathsf{C}_1, \mathsf{C}_2]^\circ$, will simply be denoted F.

We state the definitions first, then we prove that the structures are the ones that we claim they are. Given a functor $F : \mathsf{C}_1 \to \mathsf{C}_2$, define its dual $\delta F : \mathsf{C}_1 \to \mathsf{C}_2$ by

$$\delta F := \delta_2^\circ \circ F^\circ \circ \delta_1,$$

and given a natural transformation $\alpha : F \Rightarrow G$ between functors $\mathsf{C}_1 \to \mathsf{C}_2$, define its dual $\delta\alpha : F^\delta \Leftarrow G^\delta$ by

$$\delta\alpha := 1_{\delta_2^\circ} \star \alpha^\circ \star 1_{\delta_1},$$

which in components is

$$\delta\alpha_x = (\delta_2\ \alpha_{(\delta_1 x)^\circ})^\circ \qquad x \in \mathsf{C}_1. \tag{100}$$

Next we want to define a natural transformation $\eta : 1_{[\mathsf{C}_1,\mathsf{C}_2]} \Rightarrow \delta^\circ\delta$, the components of which must themselves be natural transformations

$$\eta_F : F \Rightarrow \delta^\circ\delta F \qquad\qquad F \in [\mathsf{C}_1, \mathsf{C}_2].$$

Note that $\delta^\circ\delta F = \delta_2^\circ\delta_2 F\delta_1^\circ\delta_1$. We define the component of η_F at $x \in C_1$ to be the composition

$$(\eta_F)_x : Fx \xrightarrow{F(\eta_{1,x})} F\delta_1^\circ\delta_1 x \xrightarrow{\eta_{2,F\delta_1^\circ\delta_1 x}} \delta_2^\circ\delta_2 F\delta_1^\circ\delta_1 x, \tag{101}$$

where the two morphisms involved are the components of η_1 and η_2 at the objects x and $F\delta_1^\circ\delta_1 x$, respectively.

Proposition 2.3.1. *The data (δ, η) defined above is a duality involution on $[\mathsf{C}_1, \mathsf{C}_2]$.*

Proof. We need to check the that following triangle commutes for every $F \in [\mathsf{C}_1, \mathsf{C}_2]$:

$$\begin{array}{ccc} \delta^\circ F & \xrightarrow{\eta_{\delta^\circ F}} & \delta^\circ\delta\delta^\circ F \\ & \searrow^{1_{\delta^\circ F}} & \downarrow \delta^\circ(\eta_F)^\circ \\ & & \delta^\circ F \end{array} \tag{102}$$

To check that $\delta^\circ(\eta_F)^\circ \circ \eta_{\delta^\circ F} = 1_{\delta^\circ F}$ as natural transformations, it is sufficient to check componentwise. So fix $x \in \mathsf{C}_1$. We have to show that

$$\delta^\circ Fx \xrightarrow{(\eta_{\delta^\circ F})_x} \delta^\circ\delta\delta^\circ Fx \xrightarrow{(\delta^\circ(\eta_F)^\circ)_x} \delta^\circ Fx \tag{103}$$

is equal to $(1_{\delta^\circ F})_x$. We first unpackage the left-hand half of the composition (103). By definition (101), $(\eta_{\delta^\circ F})_x = \eta_{2,\delta F\delta_1^\circ\delta_1 x} \circ \delta^\circ F(\eta_{1,x})$, which, in further detail, is

$$(\eta_{\delta^\circ F})_x : \delta_2^\circ F^\circ\delta_1 x \xrightarrow{\delta_2^\circ F^\circ\delta_1(\eta_{1,x})} \delta_2^\circ F^\circ\delta_1\delta_1^\circ\delta_1 x \xrightarrow{\eta_{2,\delta_2^\circ F^\circ\delta_1\delta_1^\circ\delta_1 x}} \delta_2^\circ\delta_2\delta_2^\circ F^\circ\delta_1\delta_1^\circ\delta_1 x. \tag{104}$$

Next we unpackage the rigth-hand half of the composite (103), i.e. $(\delta^\circ(\eta_F)^\circ)_x$. We recall that

$$(\delta^\circ(\eta_F)^\circ))_x = (\delta\eta_F)_x^\circ = (\delta\eta_F)_x)^\circ = (\delta_2(\eta_F)_{\delta_1^\circ x^\circ})^\circ$$

(using (100) for the last equality) and

$$(\eta_F)_{\delta_1^\circ x^\circ} = \eta_{2,F\delta_1^\circ\delta_1\delta_1^\circ x^\circ} \circ F(\eta_F)_{\delta_1^\circ x^\circ}$$

by definition (101). Thus

$$(\delta^\circ(\eta_F)^\circ))_x = (\delta_2(\eta_F)_{\delta_1^\circ x^\circ})^\circ = \delta_2^\circ F^\circ(\eta_{1,\delta_1^\circ x^\circ})^\circ \circ \delta_2^\circ(\eta_{2,F\delta_1^\circ\delta_1\delta^\circ x^\circ})^\circ. \tag{105}$$

Putting things together, we have that (103) is equal to the four-fold composite

$$\delta_2^\circ F^\circ(\eta_{1,\delta_1^\circ x^\circ})^\circ \circ \delta_2^\circ(\eta_{2,F\delta_1^\circ\delta_1\delta_1^\circ x^\circ})^\circ \circ \eta_{2,\delta_2^\circ F^\circ\delta_1\delta_1^\circ\delta_1 x} \circ \delta_2^\circ F^\circ\delta_1(\eta_{1,x}). \tag{106}$$

By the triangle identity for η_2, the composite of the middle two components of (106) are equal to $1_{\delta_2^\circ F^\circ\delta_1\delta_1^\circ\delta_1 x}$. And by the triangle identity for η_1, the remaining composite, namely

$$\delta_2^\circ F^\circ(\eta_{1,\delta_1^\circ x^\circ})^\circ \circ \delta_2^\circ F^\circ\delta_1(\eta_{1,x}),$$

is equal to $\delta_2^\circ F^\circ(1_{\delta_1 x}) = 1_{\delta_2^\circ F\delta_x} = 1_{\delta F x}$.

□

2.3.2. Endomorphism and automorphism categories. In the following, let (C,δ,η) be a category with duality involution.

LEMMA 2.3.2. *There is a category $\mathsf{End}(\mathsf{C})$ where*

- *objects are pairs (x,f), with $x \in \mathsf{C}$ and $f : x \to x$ is a morphism.*
- *morphisms $(x,f) \to (y,g)$ are maps $\phi : x \to y$ in C such that the following commutes:*

$$\begin{array}{ccc} x & \xrightarrow{f} & x \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ y & \xrightarrow[g]{} & y \end{array} \tag{107}$$

$\mathsf{End}(\mathsf{C})$ admits the following induced duality involution, which we call $(\underline{\delta},\underline{\eta})$: on objects $\underline{\delta}$ acts by

$$(x,f) \mapsto (\delta x, \delta f),$$

and on morphisms by

$$\phi \mapsto \delta\phi.$$

We define the components of $\underline{\eta}$ by $\underline{\eta}_{x,f} := \eta_x$.

PROOF. Clearly $\mathsf{End}(\mathsf{C})$ forms a category. To see that $\underline{\delta}$ is a contravariant functor, let $\phi : (x,f) \to (y,g)$ be a morphism in $\mathsf{End}(\mathsf{C})$. We check that $\underline{\delta}\phi : (\delta y, \delta g) \to (\delta x, \delta f)$ in $\mathsf{End}(\mathsf{C})$; the composition and identity laws for $\underline{\delta}$ are clear. By the assumption that ϕ is a morphism, the square (107) commutes. Since δ is a contravariant functor, also the following commutes

$$\begin{array}{ccc} \delta x & \xleftarrow{\delta f} & \delta x \\ {\scriptstyle\delta\phi}\uparrow & & \uparrow{\scriptstyle\delta\phi} \\ \delta y & \xleftarrow[\delta g]{} & \delta y \end{array}$$

which is precisely what is to be shown.

Now note that $\underline{\eta} : 1_{\mathsf{Aut(C)}} \Rightarrow \underline{\delta}^2$: since $\underline{\delta}^2(x,f) = (\delta^2 x, \delta^2 f)$, we find that

$$\begin{array}{ccc} \delta^2 x & \xrightarrow{\delta^2 f} & \delta^2 x \\ {\scriptstyle\eta_x}\downarrow & & \downarrow{\scriptstyle\eta_x} \\ x & \xrightarrow[f]{} & x \end{array}$$

commutes, and the naturality condition on $\underline{\eta}$, namely that all squares

$$\begin{array}{ccc} (\delta^2 x, \delta^2 f) & \xrightarrow{\delta^2\phi} & (\delta^2 y, \delta^2 g) \\ {\scriptstyle \eta_x}\downarrow & & \downarrow{\scriptstyle \eta_x} \\ (x, f) & \xrightarrow[\phi]{} & (y, g) \end{array}$$

commute, is satisfied as a consequence of the naturality of η.

Finally, the triangle identity (73) amounts to the requirement that the diagrams

(108)
$$\begin{array}{ccc} (\delta^\circ x^\circ, \delta^\circ f^\circ) & \xrightarrow{\eta_{x^\circ}} & (\delta^\circ\delta\delta^\circ x^\circ, \delta^\circ\delta\delta^\circ f^\circ) \\ & {\scriptstyle 1_{\delta^\circ x^\circ}}\searrow & \downarrow{\scriptstyle \delta^\circ(\eta_x)^\circ} \\ & & (\delta^\circ x^\circ, \delta^\circ f^\circ) \end{array}$$

commute for all $(x, f) \in \mathsf{End}(\mathsf{C})$ (in particular using $(x, f)^\circ = (x^\circ, f^\circ)$). This condition is satisfied: we already know that the morphisms involved are well-defined, and commutativity follows directly from the fact that η satisfies the triangle identify (73) for the duality δ on C. ☐

Remark 2.3.3. We may think of the objects of $\mathsf{End}(\mathsf{C})$ as "objects in C with extra structure", the extra structure being the datum of an endomorphism. The morphisms of $\mathsf{End}(\mathsf{C})$ may be viewed as "morphisms in C with extra conditions". Thus we have a canonical "forgetful functor"

$$U : \mathsf{End}(\mathsf{C}) \to \mathsf{C}$$

which acts trivially on morphisms and acts by

$$(x, f) \longmapsto x$$

on objects.

The following is a similar construction to the one above for endomorphisms, but this time with automorphisms. The twist is that we may modify the duality functor using the involution of inversion.

Lemma 2.3.4. *Let $(\mathsf{C}, \delta, \eta)$ be a category with duality. There is a category $\mathsf{Aut}(\mathsf{C})$ where*

- *objects are pairs (x, f), with $x \in \mathsf{C}$ and $f : x \to x$ is an isomorphism.*
- *morphisms $(x, f) \to (y, g)$ are maps $\phi : x \to y$ in C such that the following commutes:*

(109)
$$\begin{array}{ccc} x & \xrightarrow{f} & x \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ y & \xrightarrow[g]{} & y \end{array}$$

$\mathsf{Aut}(\mathsf{C})$ admits the following induced duality involution, which we call $(\underline{\delta}, \underline{\eta})$: on objects $\underline{\delta}$ acts by

$$(x, f) \mapsto (\delta x, (\delta f)^{-1}),$$

and on morphisms by

$$\phi \mapsto \delta\phi.$$

We define the components of $\underline{\eta}$ by $\underline{\eta}_{x,f} := \eta_x$.

PROOF. One may proceed similarly as in the proof of Lemma 2.3.2.

It is clear that $\mathsf{Aut}(\mathsf{C})$ forms a category. To verify that $\underline{\delta}$ is a contravariant functor, consider a morphism $\phi : (x, f) \to (y, g)$ in $\mathsf{Aut}(\mathsf{C})$. We check that $\underline{\delta}\phi : (\delta y, (\delta g)^{-1}) \to (\delta x, (\delta f)^{-1})$ in $\mathsf{Aut}(\mathsf{C})$ (composition and identity laws for $\underline{\delta}$ are clear). Since, by assumption, (109) commutes and δ is a contravariant functor, also the following commutes

$$\begin{array}{ccc} \delta x & \xrightarrow{(\delta f)^{-1}} & \delta x \\ {\scriptstyle \delta\phi}\uparrow & & \uparrow{\scriptstyle \delta\phi} \\ \delta y & \xrightarrow[(\delta g)^{-1}]{} & \delta y \end{array}$$

which means that $\underline{\delta}\phi$ is the desired kind of morphism.

Next we check that $\underline{\eta} : 1_{\mathsf{Aut}(\mathsf{C})} \Rightarrow \underline{\delta}^2$. Since $\underline{\delta}^2(x, f) = (\delta^2 x, \delta^2 f)$,

$$\begin{array}{ccc} \delta^2 x & \xrightarrow{\delta^2 f} & \delta^2 x \\ {\scriptstyle \eta_x}\downarrow & & \downarrow{\scriptstyle \eta_x} \\ x & \xrightarrow[f]{} & x \end{array}$$

commutes, and the naturality condition on $\underline{\eta}$, namely that all squares

$$\begin{array}{ccc} (\delta^2 x, \delta^2 f) & \xrightarrow{\delta^2\phi} & (\delta^2 y, \delta^2 g) \\ {\scriptstyle \eta_x}\downarrow & & \downarrow{\scriptstyle \eta_x} \\ (x, f) & \xrightarrow[\phi]{} & (y, g) \end{array}$$

commute, follows from the naturality of η.

Finally, the fact that $\underline{\eta}$ satisfies the triangle identity (73) is proved by an analogous argument as in Lemma 2.3.2. □

2.3.3. Linear endomorphism categories. Here we consider some constructions similar to the examples $\mathsf{End}(\mathsf{C})$ and $\mathsf{Aut}(\mathsf{C})$ from the previous two sections, but now with $\mathsf{C} = \mathsf{vect}^{\varepsilon}_{\mathbf{k}}$. We set $\mathsf{End}^{\varepsilon} = \mathsf{End}(\mathsf{vect}^{\varepsilon}_{\mathbf{k}})$.

Our discussion is initially of a rather loose nature; its purpose to is to describe general features common to a set of specific examples of categories with duality involution that we describe below. We consider categories whose objects are pairs (V, A), where V is a finite-dimensional vector space over some fixed field $\mathbf{k}$ and A is a linear endomorphism of V, possibly subject to some additional conditions. Morphisms $(V, A) \to (W, B)$ will be linear maps $T : V \to W$ such that $BT = TA$. We may sometimes tacitly impose that morphisms are only between objects where the vector spaces have the same dimension. Possibly one may need (or wish) to impose other conditions on morphisms as well.

The conditions on objects that we have in mind here are of the following kind. Fix an involutive function $f : D \to D$, where $D \subseteq \mathbf{k}$. Then we consider the category $\mathsf{End}_{f,D}$ whose objects are pairs (V, A), as above, such that $f(A)$ is a well-defined linear operator (and morphisms are defined as in $\mathsf{End}^{\varepsilon}$.) We will assume in the following that f is a rational function, i.e. of the form $f = p/q$ for polynomials p and q. For such f one might, for instance, choose D such that $q(x) \neq 0$ for all $x \in D$ and consider only those A for which $q(A)$ is invertible.

Thus, for example, if f is a polynomial on $D = \mathbf{k}$, we need not impose any restriction on the linear operators A. If $f(x) = 1/x$, then we might take $D = \mathbf{k}\backslash\{0\}$ and consider only linear operators A which are invertible, so that $f(A) = A^{-1}$ is defined.

The idea now is that the involution f may be used to induce a duality involution on $\mathsf{End}_{f,D}$. The induced duality functor δ acts on objects by

$$(V, A) \longmapsto (V^*, f(A^*))$$

and on morphisms by

$$(V, A) \xrightarrow{T} (W, B) \longmapsto (W^*, f(B^*)) \xrightarrow{T^*} (V^*, f(A^*)).$$

That the latter linear map is again a morphism follows from the fact that $BT = TA$ implies that $f(B)T = Tf(A)$ and hence $T^*f(B^*) = f(A^*)T^*$. (This commutation property is true for rational functions f). For δ to define a duality involution we also need a natural transformation $\eta : 1_{\mathsf{End}_{f,D}} \Rightarrow \delta^\circ\delta$. We take this to be defined in components either by the canonical embeddings $\eta_V : V \to V^{**}$, or by the negatives of these maps (c.f. Example 2.1.4 and Remark 2.1.5). By arguments analogous to the one given in the proofs of Lemmas 2.3.2 and 2.3.4, it follows that either of these choices of natural transformation satisfies the required triangle identity (73), i.e. that, for every object (V, A), the diagram

$$\begin{array}{ccc} (V,A)^* & \xrightarrow{\eta_{V^*}} & (V,A)^{***} \\ & \searrow^{1_{(V,A)^*}} & \downarrow{\scriptstyle (\eta_V)^*} \\ & & (V,A)^* \end{array} \tag{110}$$

commutes.

Definition 2.3.5. *Let $\mathbf{k}$ be a field, $D \subseteq \mathbf{k}$ a subset, and $f : D \to D$ an involutive function. Let $\varepsilon \in \{-, +\}$. We define $\mathsf{End}^{\varepsilon}_{f,D}$ to be the category with duality such that*

- *objects are pairs (V, A), where V ranges over finite-dimensional vector spaces (over $\mathbf{k}$) and A ranges, for given V, over linear endomorphisms of V such that $f(A)$ is well-defined.*
- *morphisms $(V, A) \to (V', A')$ are linear maps $T : V \to V'$ such that $A'T = TA$.*
- *the duality involution (δ, η) is as described above, with the components of η given by the canonical embeddings $V \to V^{**}$ if $\varepsilon = +$, or by the negatives of those maps if $\varepsilon = -$.*

CHAPTER 3

Fixed points

3.1. Definitions

Definition 3.1.1. *Let C be a category with a duality involution (δ, η). A **fixed point** (for the duality involution) is a pair (x, b), where $x \in ob(C)$ and $b : x \to (\delta x)^\circ$ is a morphism such that the following commutes:*

(111)
$$\begin{array}{ccc} x & \xrightarrow{\quad b \quad} & (\delta x)^\circ \\ & \searrow^{\eta_x} & \uparrow{\scriptstyle (\delta b)^\circ} \\ & & \delta^\circ \delta x \end{array}$$

*If b is a monomorphism, we call (x, b) a **non-degenerate** fixed point; if b is an isomorphism, we call (x, b) a **strong** fixed point; and if b is the identity, (x, b) is a **strict** fixed point.*

*Given a fixed point (x, b), we will sometimes call b a **fixed-point strucure** on x, or a **compatible form** on x (the latter term will in particular often be used in situations where b encodes a bilinear form of some kind which is compatible with the category with duality in question).*

Remark 3.1.2. In the above definition, we once again write "$(\delta x)^\circ$" to emphasize that we are thinking of that particular "δx" as an object in C and not in C°. As mentioned earlier, this is to give extra clarity to how we are viewing various objects. Below, we will again often drop this extra notation for reasons of readability; it should however always be implicitly clear from context what is meant.

To parse the diagram (111), recall that

$$\delta^\circ \delta x = (\delta \delta^\circ x^\circ)^\circ = (\delta(\delta x)^\circ)^\circ$$

and that $b \in \mathrm{Hom}_{\mathsf{C}}(x, (\delta x)^\circ)$ implies that $\delta b \in \mathrm{Hom}_{\mathsf{C}^\circ}(\delta x, \delta(\delta x)^\circ)$ and hence, by definition, $(\delta b)^\circ \in \mathrm{Hom}_{\mathsf{C}}((\delta(\delta x)^\circ)^\circ, (\delta x)^\circ) = \mathrm{Hom}_{\mathsf{C}}(\delta^\circ \delta x, (\delta x)^\circ)$. ◇

Definition 3.1.3. *Let C be a category with a duality involution (δ, η). A morphism $(x, b) \to (x', b')$ of fixed points is a morphism $f : x \to x'$ in C such that the following commutes*

(112)
$$\begin{array}{ccc} x & \xrightarrow{\ b\ } & \delta x \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle \delta f} \\ x' & \xrightarrow{\ b'\ } & \delta x' \end{array}$$

Example 3.1.4. Consider the category with duality $\mathsf{Vect}^\varepsilon_{\mathbf{k}}$ with $\varepsilon = +$, i.e. $\mathsf{C} = \mathsf{Vect}_{\mathbf{k}}$ with the standard duality functor δ and unit η be given in components by

$$\eta_V : V \longrightarrow V^{**}, \; x \longmapsto \iota_x : \xi \mapsto \xi(x).$$

A fixed point is a the data of a vector space V together with a linear map $b : V \longrightarrow V^*$ such that

$$\begin{array}{ccc} V & \xrightarrow{\ b\ } & V^* \\ & \searrow_{\eta_V} & \uparrow{\scriptstyle b^*} \\ & & V^{**} \end{array} \tag{113}$$

commutes. This says that b encodes a symmetric blinear form $V \times V \to \mathbf{k}$. Indeed, for the bilinear form B associated to b via

$$B(x, y) = b(x)(y) \qquad x, y \in V$$

we find

$$B(x, y) = b(x)(y) \stackrel{!}{=} (b^* \circ \iota_x)(y) = \iota_x(b(y)) = b(y)(x) = B(y, x).$$

If b is a *non-degenerate* fixed point, then B is a non-degenerate bilinear form, i.e.

$$B(x, y) = 0 \quad \forall y \in V \implies x = 0.$$

Given two fixed points (V, b) and (V', b'), a morphism

$$f : (V, b) \longrightarrow (V', b')$$

is a linear map $f : V \to V'$ such that

$$B'(fx, fy) = B(x, y) \qquad \forall x, y \in V,$$

where B' and B are the bilinear forms defined by b' and b, respectively. In other words, f is an isometry. △

EXAMPLE 3.1.5. Consider now the category with duality $\mathsf{Vect}_{\mathbf{k}}^{\varepsilon}$ with $\varepsilon = -$, i.e. now the unit η is, in components,

$$\eta_V : V \longrightarrow V^{**}, \ x \longmapsto -\iota_x : \xi \mapsto -\xi(x).$$

In this case, a fixed point (V, b) is the data of a *skew-symmetric* bilinear form on V. A strong fixed point corresponds to the notion of a **symplectic form** or **symplectic structure**; a non-degenerate fixed point corresponds to what is called a **weak symplectic structure**. On finite dimensional spaces, these two notions coincide.

Analogously as in the previous example, morphisms between fixed points in this example correspond to isometries. △

Given a category with duality involution $(\mathsf{C}, \delta, \eta)$, the fixed points and their morphisms form a category, denoted C^{δ}. Identity morphisms and composition are inherited from C. The full subcategory whose objects are strong fixed points will be denoted by C^{δ}_s.

PROPOSITION 3.1.6. *Let $(F, \psi) : (\mathsf{C}, \delta, \eta) \longrightarrow (\mathsf{C}', \delta', \eta')$ be an equivariant functor between categories with duality, and assume C^{δ} is non-empty. Then there is an induced functor $\hat{F} : \mathsf{C}^{\delta} \to \mathsf{C}'^{\delta'}$ between the respective categories of fixed points. On objects it acts by*

$$(x, b) \longmapsto (Fx, \psi_x^{\circ} \circ Fb)$$

and on morphisms by

$$f : x \to x' \ \longmapsto \ Ff : Fx \to Fx'.$$

If F is strongly equivariant and if the subcategory C^{δ}_s of strong fixed points is non-empty, then $\hat{F}$ restricts to a functor $\hat{F} : \mathsf{C}^{\delta}_s \to \mathsf{C}'^{\delta'}_s$.

Proof. First, we show that if (x,b) is a fixed point, then so is $(Fx, \psi^\circ_x \circ Fb)$. This is true because the following diagram commutes (and, in particular, the two outer paths through it):

$$
(114)\qquad
\begin{array}{ccccccc}
 & & F((\delta x)^\circ) & = & F\delta^\circ x & \xrightarrow{\psi^\circ_x} & \delta'^\circ F^\circ x \\
 & \nearrow^{Fb} & \uparrow{\scriptstyle F\delta b} & & \uparrow{\scriptstyle F\delta^\circ b^\circ} & & \uparrow{\scriptstyle \delta'^\circ F^\circ b^\circ} \\
Fx & \xrightarrow{F\eta_x} & F\delta^\circ\delta x & = & F\delta^\circ\delta x & \xrightarrow[\psi^\circ_{\delta x}]{} & \delta'^\circ F^\circ \delta x \\
 & & & & & & \downarrow{\scriptstyle \delta'^\circ \psi_x} \\
 & & & \searrow_{\eta'_{Fx}} & & & \delta'^\circ \delta' F x
\end{array}
$$

The upper left triangle commutes because (x,b) is a fixed point, the upper middle square commutes becuase $\delta^\circ b^\circ = \delta b$ as morphisms in C, the upper right square commutes by the naturality of ψ°, and the bottom triangle commutes by the equivariance of F (see (4.1.3)).

Note here that, clearly, if F is strongly equivariant and (x,b) is a strong fixed point, then so is $(Fx, \psi^\circ_x \circ Fb)$, since then ψ°_x and Fb are isomorphisms.

Next, we check that morphisms of fixed points are sent to morphisms of fixed points. Given a morphism $f : (x,b) \to (x',b')$, for $Ff : (Fx, \psi^\circ_x \circ Fb) \to (Fx', \psi^\circ_{x'} \circ Fb')$ to be a morphism of fixed points is the same as saying that the outer part of the following diagram commutes:

$$
(115)\qquad
\begin{array}{ccccc}
Fx & \xrightarrow{Fb} & F\delta x & \xrightarrow{\psi^\circ_x} & \delta' F x \\
\downarrow{\scriptstyle Ff} & & \uparrow{\scriptstyle F\delta f} & & \uparrow{\scriptstyle \delta' F f} \\
Fx' & \xrightarrow{Fb'} & F\delta x' & \xrightarrow{\psi^\circ_{x'}} & \delta' F x'
\end{array}
$$

But this is the case because the two subdiagrams commute: the left one because f is a morphism of fixed points, and the right one by the naturality of ψ°.

It is clear that $\hat{F}$ maps identity morphisms to identity morphisms. Also, it is compatible with composition, since we can stack diagrams of the kind (115). Thus $\hat{F}$ is a functor, as was to be shown. □

3.2. Examples

3.2.1. Groups. Let G be a group, and G the associated category with a single object "$*$", c.f. Section 2.2.2. Suppose G is equipped with the duality involution where $\delta = (-)^{-1}$ and the unit η is such that its single component is an element $a \in Z(G)$.

What is a fixed point in this case? A fixed point is $(*, b)$, where $b \in G$ is such that

$$
(116)\qquad
\begin{array}{ccc}
* & \xrightarrow{b} & * \\
 & \searrow_{a} & \uparrow{\scriptstyle b^{-1}} \\
 & & *
\end{array}
$$

commutes. In other words, fixed points correspond to elements of G such that $b^2 = a$. In particular, if $a = 1_G$, then fixed points are the elements of G of order 2.

3.2.2. Vector space categories. We consider the duality involution defined in Example 2.2.3 for representations of a finite group G. If $\eta : V \to V^{**}$ is the usual embedding, then a fixed point structure on a representation ψ on V corresponds to a symmetric bilinear form $B : V \to V^*$ on V which is also a morphism of representation $\psi \to \psi^*$. This is the same as saying that B is such that $B(\psi(g)v, \psi(g)w) = B(v, w)$ for all $v, w \in V$, i.e. ψ maps G to the subgroup of $\mathrm{Aut}(G)$ consisting of automorphisms which preserve B.

3.2.3. Dagger categories. Given a dagger category $(\mathsf{C}, \dagger)$, the fixed points of its duality involution are pairs (x, f) where f is an endomorphism of x satisfying $f^\dagger = f$, i.e. f is a self-adjoint endomorphism of x.

EXAMPLE 3.2.1. In the case of the dagger category hilb from Example 2.2.4, a fixed point corresponds to a Hilbert space $(H, \langle , \rangle)$ together with a self-adjoint linear map $A : H \to H$. Indeed, let $b : H \to \overline{H}^*$ be the map associated with the inner product on H. A fixed point is a linear map $A : H \to H$ such that $A^\dagger = A$; in other words $b^{-1}A^*b = A$, which means precisely that $\langle Av, w\rangle = \langle v, Aw\rangle$ for all $v, w \in H$.

EXAMPLE 3.2.2. In the case of the dagger category Rel from Example 2.2.5, a fixed point corresponds to a set x together with an endorelation $r \in x \times x$ such that

$$\begin{array}{ccc} x & \xrightarrow{\;r\;} & x \\ & \underset{Id_x}{\searrow} & \big\uparrow{\scriptstyle r^\dagger} \\ & & x \end{array} \tag{117}$$

commutes. In other words, such that $r^\dagger = r$. Thus, fixed points in this example are precisely those endorelations which are *symmetric* relations.

3.2.4. Pontryagin duality. Consider the Pontryagin duality involution on finite abelian groups from Section 2.2.6 . Similar to the standard example in vectors spaces (Example 3.1.4), a fixed point structure $b : G \longrightarrow \widehat{G}$ corresponds to a bilinear pairing

$$G \times G \longrightarrow \mathbb{R}/\mathbb{Z} \tag{118}$$

which is symmetric.

3.2.5. Powerset duality. Consider the duality involution on the category Set given by the powerset functor discussed in Section 2.2.7. A fixed point in this case is a set X equipped with a function $b : X \to \Omega^X$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\;b\;} & \Omega^X \\ & \underset{\eta_X}{\searrow} & \big\uparrow{\scriptstyle (-)\circ b} \\ & & \Omega^{(\Omega^X)} \end{array} \tag{119}$$

commutes; that is, for any $x, y \in X$, we must have $b(x)(y) = b(y)(x)$. Let's look at what this translates to if we think of Ω^X in terms of the set of all subsets of X. In this case, the lower path through the diagram is the function

$$X \longrightarrow \mathcal{P}(X),\ x \longmapsto \{y \in X \mid x \in b(y)\}.$$

Thus, $b : X \to \mathcal{P}(X)$ is a fixed point if and only if $b(x) = \{y \in X \mid x \in b(y)\}$, which is equivalent to saying that b has the property that

$$y \in b(x) \Leftrightarrow x \in b(y). \tag{120}$$

In Set, the functor $(-)^X$ is right-adjoint to the functor $X \times (-)$. In particular, we have a bijection

$$\mathsf{Set}(X, \Omega^X) \simeq \mathsf{Set}(X \times X, \Omega), \tag{121}$$

and functions $X \times X \to \Omega$, in turn, correspond to subsets of $X \times X$, i.e. endorelations on X. Under the bijection from functions $b : X \to \Omega^X$ to relations $R_b \subseteq X \times X$, fixed points b correspond to endorelations which are *symmetric*. Thus, fixed points for the powerset duality on Set amount to the same thing as the fixed points in Rel with respect to its dagger structure (Example 3.2.2).

3.2.6. Endomorphism categories. Let $(\mathsf{C}, \delta, \eta)$ be a category with duality, and $(\mathsf{End}(C), \underline{\delta}, \eta)$ the associated category with duality as defined in Section 2.3.2. Spelling out the definition of a fixed point, we see that a fixed point in $\mathsf{End}(C)$ is a pair $((x, f), b)$, where $f : x \to x$ and $b : x \to \delta x$ are morphisms in C such that the diagrams

$$\begin{array}{ccc} x & \xrightarrow{\;b\;} & \delta x \\ & {\scriptstyle \eta_x}\searrow & \uparrow{\scriptstyle \delta b} \\ & & \delta\delta x \end{array} \qquad \text{and} \qquad \begin{array}{ccc} x & \xrightarrow{\;b\;} & \delta x \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \delta f} \\ x & \xrightarrow[\;b\;]{} & \delta x \end{array} \tag{122}$$

commute.

3.2.7. Automorphism categories. Once again let $(\mathsf{C}, \delta, \eta)$ be a category with duality, and let $(\mathsf{Aut}(C), \underline{\delta}, \eta)$ be the associated category with duality as defined in Section 2.3.2. By definition, a fixed point in $\mathsf{End}(C)$ is a pair $((x, f), b)$, where $f : x \to x$ and $b : x \to \delta x$ are morphisms in C such that the diagrams

$$\begin{array}{ccc} x & \xrightarrow{\;b\;} & \delta x \\ & {\scriptstyle \eta_x}\searrow & \uparrow{\scriptstyle \delta b} \\ & & \delta\delta x \end{array} \qquad \text{and} \qquad \begin{array}{ccc} x & \xrightarrow{\;b\;} & \delta x \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (\delta f)^{-1}} \\ x & \xrightarrow[\;b\;]{} & \delta x \end{array} \tag{123}$$

commute.

Observe that the commutativity of the second diagram says in fact that f is an automorphism of the fixed point (x, b) (compare with Definition 3.1.3). This leads to the following

PROPOSITION 3.2.3. *Let $(\mathsf{C}, \delta, \eta)$ be a category with duality. Then $\mathsf{Aut}(C)^{\underline{\delta}}$ and $\mathsf{Aut}(C^{\delta})$ are isomorphic categories.*

PROOF. As noted above, objects in $\mathsf{Aut}(C)^{\underline{\delta}}$ are pairs $((x, f), b)$ where (x, f) is an object of $\mathsf{Aut}(C)$, (x, b) is a fixed point of $(\mathsf{C}, \delta, \eta)$, and f is an automorphism of this fixed point.

A morphism $((x, f), b) \to ((x', f'), b')$ in $\mathsf{Aut}(C)^{\underline{\delta}}$ is a morphism γ in $\mathsf{Aut}(C)$ such that

$$\begin{array}{ccc} (x, f) & \xrightarrow{\;b\;} & (\delta x, \delta f^{-1}) \\ {\scriptstyle \gamma}\downarrow & & \uparrow{\scriptstyle \delta\gamma} \\ (x', f') & \xrightarrow{\;b'\;} & (\delta x', \delta f'^{-1}) \end{array} \tag{124}$$

commutes (in $\mathsf{Aut}(\mathrm{C})$). This is equivalent to saying that the following two squares commute in C:

$$\begin{CD} x @>b>> \delta x \\ @V{\gamma}VV @AA{\delta\gamma}A \\ x' @>b'>> \delta x' \end{CD} \qquad \text{and} \qquad \begin{CD} x @>f>> x \\ @V{\gamma}VV @VV{\gamma}V \\ x' @>f'>> x' \end{CD} \tag{125}$$

On the other hand, objects in $\mathsf{Aut}(C^\delta)$ are pairs $((x,b),f)$ where (x,b) is a fixed point and $f:(x,b)\to(x,b)$ is an isomorphism, i.e. f is an isomorphism $x\to x$ such that

$$\begin{CD} x @>b>> \delta x \\ @V{f}VV @AA{\delta f}A \\ x' @>b'>> \delta x' \end{CD} \tag{126}$$

commutes. As noted above, this diagram is the same as the second diagram in (123), so objects $((x,b),f)$ in $\mathsf{Aut}(C^\delta)$ are essentially the same thing as objects $((x,f),b)$ in $\mathsf{Aut}(C)^{\underline{\delta}}$, the only difference begin the bracketing.

A morphism $((x,b),f)\to((x',b'),f')$ in $\mathsf{Aut}(C^\delta)$ is a morphism $\gamma:(x,b)\to(x',\theta')$ in C^δ which intertwines f and f', which is precisely the condition that the two squares (125) commute. Thus morphisms in $\mathsf{Aut}(C^\delta)$ are the same, on the nose, as morphisms in $\mathsf{Aut}(C)^{\underline{\delta}}$. □

3.2.8. Linear endomorphism categories. In this section we consider categories with duality of the kind discussed in Section 2.3.3, i.e. categories of the kind $\mathsf{End}^{\varepsilon}_{f,D}$, which were variations on $\mathsf{End}(\mathrm{vect}^{\varepsilon}_{\mathbf{k}})$.

EXAMPLE 3.2.4. Let $D=\mathbf{k}$, $f=\mathrm{id}_{\mathbf{k}}$, and $\varepsilon=\pm$. Then $\mathsf{End}^{\varepsilon}_{f,D}=\mathsf{End}(\mathrm{vect}^{\pm}_{\mathbf{k}})$, i.e. we assume $\mathrm{vect}_{\mathbf{k}}$ is equipped with the standard duality functor δ and the components of the unit η for δ are the canonical embeddings $V\to V^{**}$ or their negatives (depending on the sign given by ε).

From Section 3.2.6 we know that a fixed point in $(\mathsf{End}(\mathrm{vect}_{\mathbf{k}}),\underline{\delta},\eta)$ corresponds to an object $V\in\mathsf{vect}_{\mathbf{k}}$ and morphisms $X:V\to V$ and $b:V\to V^*$ such that

$$\begin{array}{ccc} V & \xrightarrow{\;b\;} & V^* \\ & {\scriptstyle\eta_V}\searrow & \uparrow{\scriptstyle b^*} \\ & & V^{**} \end{array} \qquad \text{and} \qquad \begin{CD} V @>b>> V^* \\ @V{X}VV @VV{X^*}V \\ V @>>b> V^* \end{CD} \tag{127}$$

commute. The commutativity of the first diagram says that b encodes a ε-symmetric bilinear form B on V, i.e. a form such that

$$B(v,w)=\varepsilon B(w,v) \qquad \forall v,w\in V^1.$$

The commutativity of the second diagram says that X is self-adjoint with respect to B, i.e. that

$$B(Xv,w)=B(v,Xw) \qquad \forall v,w\in V.$$

[1]We are slightly overloading the symbol ε here, since initially it was introduced as a "formal" symbol, and now we are also using it as a stand-in for the scalars ±1 in the ground field $\mathbf{k}$.

EXAMPLE 3.2.5. If, in the previous example, we take $f = -\mathrm{id}_{\mathbf{k}}$ instead of $f = \mathrm{id}_{\mathbf{k}}$, then a fixed point corresponds to an object $V \in \mathsf{vect}_{\mathbf{k}}$ and morphisms $X : V \to V$ and $b : V \to V^*$ such that

$$\begin{array}{ccc} V & \xrightarrow{b} & V^* \\ & \searrow^{\eta_V} & \uparrow{\scriptstyle b^*} \\ & & V^{**} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} V & \xrightarrow{b} & V^* \\ {\scriptstyle X}\downarrow & & \downarrow{\scriptstyle -X^*} \\ V & \xrightarrow[b]{} & V^* \end{array} \tag{128}$$

commute. In other words, in this case, X is *skew*-selfadjoint with respect to the ε-symmetric bilinear form encoded by b, i.e.

$$B(Xv, w) = -B(v, Xw) \qquad \forall v, w \in V.$$

EXAMPLE 3.2.6. Let $D = \mathbf{k}\backslash\{0\}$, $f(x) = 1/x$, and $\varepsilon = \pm$. Then $\mathsf{End}^{\varepsilon}_{f,D} = \mathsf{Aut}(\mathrm{vect}^{\pm}_{\mathbf{k}})$.

From Section 3.2.7 we know that a fixed point in $(\mathsf{Aut}(\mathrm{vect}_{\mathbf{k}}), \underline{\delta}, \eta)$ corresponds to an object $V \in \mathsf{vect}_{\mathbf{k}}$ and morphisms $X : V \to V$ and $b : V \to V^*$ such that

$$\begin{array}{ccc} V & \xrightarrow{b} & V^* \\ & \searrow^{\eta_V} & \uparrow{\scriptstyle b^*} \\ & & V^{**} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} V & \xrightarrow{b} & V^* \\ {\scriptstyle X}\downarrow & & \uparrow{\scriptstyle X^*} \\ V & \xrightarrow[b]{} & V^* \end{array} \tag{129}$$

commute. The commutativity of the first diagram says, again, that b encodes a ε-symmetric bilinear form B on V; the commutativity of the second diagram says that X is an isometry with respect to B, i.e. that

$$B(Xv, Xw) = B(v, w) \qquad \forall v, w \in V.$$

EXAMPLE 3.2.7. If we take $f(x) = -1/x$ instead of $f(x) = 1/x$ in the previous example, then a fixed point involves a ε-symmetric bilinear form B on a vector space V and a *skew*-isometry (or anti-isometry) X with respect to that form, i.e.

$$B(Xv, Xw) = -B(v, w) \qquad \forall v, w \in V.$$

3.3. Subobjects of fixed points

Fix a category with duality $(\mathsf{C}, \delta, \eta)$. Assume in this section that C has a zero object, i.e. an object 0 which is both initial and terminal. In particular, between any two objects x and y there is then the notion of the zero arrow $x \to 0 \to y$ from x to y. We also assume that for any zero object $0 \in \mathsf{C}$, also $\delta 0$ is a zero object in C°. This is for example the case when we are in the setting of additive categories.

Let $b : x \to \delta x$ be a fixed point. We will define a notion of orthogonality for subobjects of x.

The motivating example that we have in mind is the case where $x = V$ is a finite-dimensional vector space. Suppose V is equipped with a bilinear form $b : V \to V^*$. Given subspaces U and W of V, we say that U is **left-orthogonal** to W (and W **right-orthogonal** to U) if

$$b(u)(w) = 0 \quad \forall u \in U \ w \in W. \tag{130}$$

We denote this by $U \perp W$. If b is symmetric or skew-symmetric, then orthogonality is a symmetric relation, i.e. $U \perp W \Leftrightarrow W \perp U$. A way to rephrase (130) is to say that

$$U \xrightarrow{i_U} V \xrightarrow{b} V^* \xrightarrow{i_W^*} W^* \tag{131}$$

is the zero morphism (where i_U and i_W are the inclusion maps).

Definition 3.3.1. *Let $b : x \to \delta x$ be a fixed point in C, and let $i : u \to x$ and $j : w \to x$ be subobjects. We say that i is* ***orthogonal*** *to j (or that u is* ***orthogonal*** *to w) if the composite*

$$u \xrightarrow{i} x \xrightarrow{b} \delta x \xrightarrow{\delta j} \delta w \tag{132}$$

is the zero morphism. In this case we write $i \perp j$ (or $u \perp w$).

Remark 3.3.2. Note that if $i \simeq i'$ as subobjects, and if $i \perp j$, then also $i' \perp j$.

Lemma 3.3.3. *Orthogonality of subobjects of (x, b) is a symmetric relation.*

Proof. Assume that $i \perp j$. We'll show that $j \perp i$. Consider the diagram

$$\begin{array}{ccccccc} \delta u & \xleftarrow{\delta i} & \delta x & \xleftarrow{\delta b} & \delta\delta x & \xleftarrow{\delta\delta j} & \delta\delta w \\ & & & \nwarrow^{b} & \uparrow{\scriptstyle \eta_x} & & \uparrow{\scriptstyle \eta_w} \\ & & & & x & \xleftarrow[j]{} & w \end{array} \tag{133}$$

It is commutative: the middle triangle is because (x, b) is a fixed point, and the right-hand square is by the naturality of η. The top horizontal composition is the zero arrow, because it is the image under δ of (132), and δ is assumed to preserve zero objects. Thus also the precomposition with η_w is the zero arrow. By the commutativity of the diagram, it follows then that also

$$\delta u \xleftarrow{\delta i} \delta x \xleftarrow{b} \delta\delta x \xleftarrow{j} w \tag{134}$$

is the zero arrow, which means $j \perp i$. □

The following is straightforward check.

Lemma 3.3.4. *Let $i : u \to x$, $j : w \to x$ and $k : v \to x$ be subobjects of the fixed point (x, b). If $j \perp i$ and $k \leq j$, then $k \perp i$.*

Definition 3.3.5. *Let $i : u \to x$ be a subobject of the fixed point (x, b). An* ***orthogonal*** *of $i : u \to x$ is a subobject $i^\perp : u^\perp \to x$ such that*

- *$i^\perp$ is orthogonal to i, and*
- *if $i' \in \mathsf{Sub}(x)$ is orthogonal to i then $i' \leq i^\perp$.*

Lemma 3.3.6. *Orthogonals are essentially unique: if $u \simeq w$ as subobjects, then $u^\perp \simeq w^\perp$ as subobjects.*

Proof. If $u \simeq w$, then $u \perp w^\perp$ and $w \perp u^\perp$. This implies $w^\perp \leq u^\perp$ and $u^\perp \leq w^\perp$. □

Remark 3.3.7. Because of the previous lemma, we will often refer to "the orthogonal", even though this only has meaning up to isomorphism of subobjects.

Lemma 3.3.8. *Let $i : u \to x$ be a subobject of the fixed point (x, b). If the orthogonal $u^\perp$ exists, it is the coproduct of all subobjects $j : w \to x$ which are orthogonal to i.*

Proof. By Lemma 3.3.4 above, if j is orthogonal to i, then $j \leq i^\perp$. We take this as the inclusion map for the coproduct (there is no other choice anyway). To check the universal property of the coproduct, suppose that $k : v \to x$ is a subobject such that $j \leq k$ for all j which are othogonal to i. Since by definition $i^\perp$ is orthogonal to i, this means that also $i^\perp \leq k$, which is the desired universal map. □

PROPOSITION 3.3.9. *Assume the fixed point* (x, b) *is such that in* $\mathsf{Sub}(x)$ *all orthogonals exist. Furthermore, assume that for each subobject, a particular orthogonal is chosen. Then the operation of "taking the orthogonal" induces a duality involution*

$$(-)^{\perp} : \mathsf{Sub}(x) \longrightarrow \mathsf{Sub}(x)^{\circ}. \tag{135}$$

PROOF. First we specify $(-)^{\perp} : \mathsf{Sub}(x) \longrightarrow \mathsf{Sub}(x)^{\circ}$. Given a morphism $f : i \to j$, we will show that $j^{\perp} \leq i^{\perp}$, and in this case we define $f^{\perp}$ to be the unique morphism "$\leq$". Consider the diagram

$$\begin{array}{ccccccc} u & \xrightarrow{j} & x & \xrightarrow{b} & \delta x & \xrightarrow{\delta j^{\perp}} & \delta u \\ {\scriptstyle f}\uparrow & \nearrow_{i} & & & & & \\ w & & & & & & \end{array} \tag{136}$$

The left triangle commutes since $f : i \to j$ is a morphism in $\mathsf{Sub}(x)$. Since $j \perp j^{\perp}$, the top horizontal composite is the zero arrow. This implies that the path from from w to δu is also the zero arrow, which means that $i \perp j^{\perp}$. It follows that $j^{\perp} \leq i^{\perp}$. That $(-)^{\perp}$ is indeed functorial follows automatically from the fact that $\mathsf{Sub}(x)$ is a thin category.

It remains to specify the unit η for the duality involution. Since $\mathsf{Sub}(x)$ is thin, we just need to show that $i \leq (i^{\perp})^{\perp}$ for every subobject i. We have both $i \perp i^{\perp}$ and $i^{\perp} \perp (i^{\perp})^{\perp}$. By definition $(i^{\perp})^{\perp}$ contains all subobjects orthogonal to $i^{\perp}$, so it follows that $i \leq (i^{\perp})^{\perp}$. □

For the remainder of this section, we assume that for any fixed point (x, b), all orthogonals of subobjects exist, and also all finite products and coproducts of subobjects. The notation $i \wedge j$ and $i \vee j$ will denote the product and coproduct, respectively, of subobjects i and j.

DEFINITION 3.3.10. *The* ***radical****,* $\mathrm{rad}(x)$*, of the fixed point* (x, b) *is the orthogonal of the top subobject* $1 : x \to x$*. More generally, the* ***radical of a subobject*** $i : u \to x$ *is*

$$\mathrm{rad}(i) = i^{\perp} \wedge i. \tag{137}$$

LEMMA 3.3.11. *If* (x, b) *is a non-degenerate fixed point, then* $\mathrm{rad}(x)$ *is the bottom subobject* 0*.*

PROOF. Let $r : \mathrm{rad}(x) \to x$ be the radical of x. Then by definition, r is orthogonal to 1, i.e.

$$\mathrm{rad}(x) \xrightarrow{r} x \xrightarrow{b} \delta x \xrightarrow{\delta(1_x)} \delta x \tag{138}$$

is the zero morphism. Since $\delta 1_x = 1_{\delta x}$, this implies that $br = 0$. On the other hand, the bottom subobject also satisfies $b0 = 0$. By the right-cancellation property of the monomorphism b, this implies that $r = 0$. □

In Part 2 of the thesis we will be interested in certain special kinds of subobjects of fixed points. Some of the definitions work in full generality, so we formulate them here.

DEFINITION 3.3.12. *Let* (x, b) *be a fixed point. A subobject* $i : u \to x$ *is*

- ***isotropic*** *if* $i \leq i^{\perp}$*;*
- ***coisotropic*** *if* $i^{\perp} \leq i$*;*
- ***lagrangian*** *if* $i = i^{\perp}$*.*

3.4. Endomorphisms of strong fixed points

Let $(\mathsf{C}, \delta, \eta)$ be a category with duality. Given a strong fixed point (x, b), the monoid $\mathrm{End}(x)$ inherits a $*$-structure as follows. For $f \in \mathrm{End}(x)$, we define the **transpose**, f^t, to be the composite

$$x \xrightarrow{b} \delta^\circ x^\circ \xrightarrow{\delta^\circ f^\circ} \delta^\circ x^\circ \xrightarrow{b^{-1}} x \tag{139}$$

Lemma 3.4.1. *Given a strong fixed point (x, b), the operation of transpose defines a $*$-structure on* $\mathrm{End}(x)$.

Proof. Indeed,

$$\begin{aligned}(gf)^t =& b^{-1}\delta^\circ(gf)^\circ b = b^{-1}\delta^\circ(f^\circ g^\circ)b \\ &= b^{-1}\delta^\circ f^\circ \delta^\circ g^\circ b = b^{-1}\delta^\circ f^\circ b b^{-1}\delta^\circ g^\circ b = f^t g^t.\end{aligned}$$

□

Remark 3.4.2. The above does not fully use that (x, b) is a fixed point; it only uses that $b : x \to \delta^\circ x^\circ$ is an isomorphism.

Remark 3.4.3. If (x, b) is a non-degenerate fixed point (i.e. not necessarily strong), we can define the transpose of an endomorphism $f \in \mathrm{End}(x)$ – *if it exists* – to be the unique map f^t such that

$$(\delta^\circ f^\circ) \circ b = b \circ f^t. \tag{140}$$

Uniqueness is guaranteed by the right-cancellation property of b.

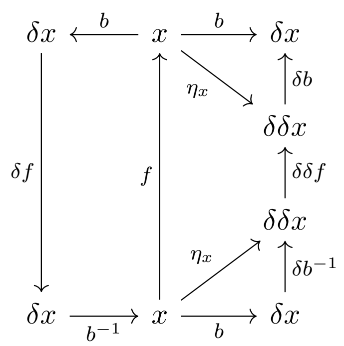
Example 3.4.4. Let C be the category of finite-dimensional $\mathbf{k}$-vector spaces, let $b : V \to V^*$ be an isomorphism, and $B : V \times V \to \mathbf{k}$ the corresponding bilinear form. Given an endomorphism $f \in \mathrm{End}(V)$, the transpose f^t is the usual notion, i.e. it is the unique endomorphism satisfying

$$B(fv, w) = B(v, f^t w) \qquad \forall\, v, w \in V. \tag{141}$$

Lemma 3.4.5. *Let (x, b) be a strong fixed point. If f is an endomorphism of (x, b), then f^t is, too.*

Proof. We need to show that $b = \delta f^t \circ b \circ f^t$. This is done if we show that the following diagram commutes:

(142)



To see that this does in fact commute, note that the triangular cells do because b is a fixed point, the right-hand trapozoidal cell is the naturality square for η, and the left-hand rectangular cell commutes by the assumption that f is an endomorphism of (x, b). □

LEMMA 3.4.6. *Let* (x, b) *be a strong fixed point, and* $f \in End(x)$*. Suppose that* f *has a kernel* $k \stackrel{i_k}{\to} x$ *and that* f^t *has an image* y *with* $f = x \stackrel{p_y}{\to} y \stackrel{i_y}{\to} x$*, where* p_y *is epic and* i_y *monic. Then* $Ker(f) \perp Im(f^t)$*.*

PROOF. We need to show that $k \stackrel{i_k}{\to} x \stackrel{b}{\to} \delta \stackrel{\delta i_y}{\to} \delta y$ is zero. For this it is sufficient to show that

$$k \stackrel{i_k}{\to} x \stackrel{b}{\to} \delta x \stackrel{\delta i_y}{\to} \delta y \stackrel{\delta p_y}{\to} \delta x \stackrel{b^{-1}}{\to} x \tag{143}$$

is zero, since δp_y is monic (and so also $b^{-1} \circ p_y$), and so we can use the cancellation property of monics. Now we observe that (143) is in fact equal to

$$0 = k \stackrel{i_k}{\to} x \stackrel{f}{\to} x, \tag{144}$$

because

$$b^{-1} \circ \delta i_y \circ \delta p_y \circ b = b^{-1} \circ \delta f^t \circ b = b^{-1} \circ \delta b \circ \delta\delta f \circ \delta b^{-1} \circ b = \eta^{-1} \circ \delta\delta f \circ \eta = f.$$

□

CHAPTER 4

Additive categories with duality

As mentioned in the introduction to this thesis, we include material on duality involutions on additive categories, in order to use the approach of Scharlau et al. and Sergeichuk for geometric classification problems. The main tools are the hyperbolization construction, described in Section 4.2, together with Proposition 4.4.16, the proof of which is based on Lemma 2 in [**Ser87**]. The ideas and results in this chapter will be used explicitly and implicitly throughout Part 2 of this thesis.

4.1. Duality involutions, fixed points

In this section we discuss duality involutions (and their fixed points) in the context of additive categories. As mentioned in Section 1.5, additive categories form a 2-category AddCat. The definition of an *additive* duality involution and of a morphism of *additive* categories with duality are nearly identical with the "original" definition. The only difference is that now we use the appropriate definition of "1-morphism" and "2-morphism" in AddCat in place of the corresponding notions used before (where the relevant 2-category was simply Cat). Despite the repetitiveness, we spell out the initial definitions for additive categories.

Definition 4.1.1. *An* ***additive duality involution*** *on an additive category C is a pair (δ, η), where $\delta : \mathsf{C} \to \mathsf{C}^\circ$ is an additive functor and η is a natural transformation $1_{\mathsf{C}} \overset{\eta}{\Rightarrow} \delta^\circ\delta$ such that, for all $x \in ob(\mathsf{C})$,*

(145)
$$\begin{array}{ccc} \delta^\circ x^\circ & \xrightarrow{\eta_{\delta^\circ x^\circ}} & \delta^\circ\delta\delta^\circ x^\circ \\ & {\scriptstyle 1_{\delta^\circ x^\circ}} \searrow & \downarrow {\scriptstyle \delta^\circ(\eta_x)^\circ} \\ & & \delta^\circ x^\circ \end{array}$$

commutes. We call an additive category equipped with an additive duality involution an ***additive category with duality****.*

As in the original definition of duality involution, we use the adjectives "strong" or "strict", respectively, to indicate if η is a natural isomorphism or an equality.

Example 4.1.2. On the additive category $\mathsf{vect}_{\mathbf{k}}$, the usual duality functor with $\delta V = \mathrm{Hom}(V, \mathbf{k}) = V^*$ is additive. Indeed,

$$(\varphi^{\delta}_{V,W})^\circ : V^* \oplus W^* \longrightarrow (V \oplus W)^* \tag{146}$$

given by

$$(\xi, \eta) \longmapsto [(v, w) \mapsto \xi(v) + \eta(v)] \tag{147}$$

defines the universal map from the biproduct $V^* \oplus W^*$, i.e. the diagram

(148)
$$\begin{array}{ccccc} V^* & \xrightarrow{i_{V^*}} & V^* \oplus W^* & \xleftarrow{i_{W^*}} & W^* \\ & {\scriptstyle (p_V)^*} \searrow & \downarrow {\scriptstyle (\varphi^{\delta}_{V,W})^{\circ}} & \swarrow {\scriptstyle (p_W)^*} & \\ & & (V \oplus W)^* & & \end{array}$$

commutes.

DEFINITION 4.1.3. *A morphism, or* ***equivariant functor****,* $(\mathsf{C}, \delta, \eta) \longrightarrow (\mathsf{D}, \delta', \eta')$ *between additive categories with duality is a pair* (F, ψ)*, where* $F : \mathsf{C} \to \mathsf{D}$ *is an additive functor, and* ψ *is a natural transformation*

$$\begin{array}{ccc} \mathsf{C} & \xrightarrow{F} & \mathsf{D} \\ {\scriptstyle \delta}\downarrow & \swarrow\!\!\!\swarrow {\scriptstyle \psi} & \downarrow{\scriptstyle \delta'} \\ \mathsf{C}^{\circ} & \xrightarrow[F^{\circ}]{} & \mathsf{D}^{\circ} \end{array}$$

such that

(149)
$$(\delta'^{\circ}\psi) \circ \eta' F = (\psi^{\circ}\, \delta) \circ F\eta.$$

Again, adjectives "strong" and "strict" may indicate the "strength" of ψ. Composition works in the same way as for "ordinary" equivariant functors, see (83).

For natural transformations between morphisms of additive categories with duality, the definition is absolutely identical with Definition 2.1.10; in particular vertical and horizontal composition works in the same way.

Additive categories with duality, together with their morphisms and natural transformations, form a 2-category which we denote by $\mathsf{dAddCat}$.

Given an additive category with duality $(\mathsf{C}, \delta, \eta)$, the notions of a **fixed point** and a **morphism of fixed points** are defined in the same was as in the "ordinary" case, i.e. as in Definition 3.1.1 and Definition 3.1.3. In particular, Proposition 3.1.6 also applies directly here in the context of additive categories, i.e. equivariant functors induce functors between the corresponding categories of fixed points. The category of fixed points of a given additive category with duality need *not* however be additive. We do have the following.

PROPOSITION 4.1.4. *Let* $(\mathsf{C}, \delta, \eta)$ *be an additive category with duality. Then the category* C^{δ} *of fixed points has all finite coproducts, and so does the subcategory* C^{δ}_s *of strong fixed points.*

In particular, C^{δ}_s *has an initial object, so* C^{δ}_s *and* C^{δ} *are non-empty.*

PROOF. First we show that C^{δ} has an initial object. Our candidate is $(z, 0)$, where z is a zero object in C and $z \xrightarrow{0} \delta z$ is the zero morphism. Note that since δ is additive, δz is also a zero object. That $(z, 0)$ is a fixed point follows from the fact that both paths through the diagram

$$\begin{array}{ccc} z & \xrightarrow{0} & \delta z \\ & {\scriptstyle \eta_z}\searrow & \uparrow{\scriptstyle \delta 0 = 0} \\ & & \delta\delta z \end{array}$$

must give the zero morphism $z \to \delta z$. Moreover, $(z, 0)$ is in fact a strong fixed point: the involved map 0 is an isomorphism, because it is the unique map between the zero objects z and δz in C. To check that $(z, 0)$ is initial, let (x, f) be some other fixed point. Since

z is initial in C, we have a unique morphism $z \to x$ in C, which is necessarily the zero morphism. And the zero morphism is indeed a morphism of fixed points, since

$$\begin{CD} z @>0>> x \\ @V{0}VV @VV{f}V \\ \delta z @<<{\delta 0 = 0}< \delta x \end{CD} \tag{150}$$

commutes.

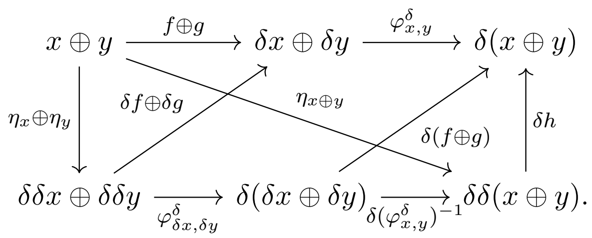
Next we wish to define the binary coproduct in C^δ. Given fixed points (x, f) and (y, g), we set

$$(x, f) \sqcup (y, g) := (x \oplus y, \varphi^\delta_{x,y} \circ (f \oplus g)), \tag{151}$$

and we take the inclusion maps i_x and i_y for the biproduct $x \oplus y$ as candidates for the inclusion maps of the coproduct (151). For notational convenience, let $h := \varphi^\delta_{x,y} \circ (f \oplus g)$. (*A priori*, $\varphi^\delta_{x,y}$ is a morphism $\delta(x \oplus x) \to \delta x \oplus \delta y$ in C°; thus, above, we are technically/implicitely using $(\varphi^\delta_{x,y})^\circ$, but we call it $\varphi^\delta_{x,y}$ as well.)

To see that (151) defines a fixed point, note that in the following diagram

(152)

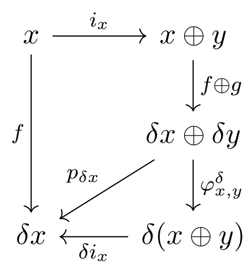


the upper left triangle commutes since (x, f) and (y, g) are fixed points, the middle parallelogram commutes by the binaturality of the coherence isomorphisms, the lower right triangle commutes by the definition of h and the functoriality of δ, and the lower left triangle commutes by (60). This all implies that the upper right triangle commutes, which is what is needed to be shown.

Note that $\varphi^\delta_{x,y} \circ (f \oplus g)$ is an isomorphism if f and g are, so $(x, f) \sqcup (y, g)$ is a strong fixed-point when (x, f) and (y, g) are.

Now we check that the inclusion maps i_x and i_y are also morphisms in C^δ. Consider i_x (the proof for i_y is completely analogous). We need that the outer paths through the diagram

(153)



define the same morphism. This is the case since the two subdiagrams commute. The upper diagram does because

$$p_{\delta x} \circ (f \oplus g) \circ i_x = p_{\delta x} \circ i_{\delta x} \circ f = f,$$

where for the first equality we use that $f \oplus g$ is the copairing out of $x \oplus y$ viewed as a coproduct, and for the second equality we use that $p_{\delta x} \circ i_{\delta x} = 1_{\delta x}$. The lower subdiagram is the opposite diagram of the following diagram in C°

(154)
$$\begin{array}{ccc} & & \delta x \oplus \delta y \\ & \overset{i_{\delta x}}{\nearrow} & \uparrow{\scriptstyle \varphi^{\delta}_{x,y}} \\ \delta x & \xrightarrow[\delta i_x]{} & \delta(x \oplus y) \end{array}$$

since $(i_{\delta x})^{\circ} = p_{\delta x}$ by virtue of the fact that the definition of a biproduct is self-dual. (Beware, again, that we are not distinguishing between $\varphi^{\delta}_{x,y}$ and $(\varphi^{\delta}_{x,y})^{\circ}$ in our notation here, nor between δi_x and $(\delta i_x)^{\circ}$.) That (154) commutes follows from the fact that $(\varphi^{\delta}_{x,y})^{-1}$ is the map, guaranteed by the universal property of the coproduct $\delta x \oplus \delta y$, such that $(\varphi^{\delta}_{x,y})^{-1} \circ i_{\delta x} = \delta i_x$. Thus $i_{\delta x} = \varphi^{\delta}_{x,y} \circ \delta i_x$.

Finally, we need to check that $(x, f) \sqcup (y, g) = (x \oplus y, \varphi^{\delta}_{x,y} \circ (f \oplus g))$ satisfies the universal property of the coproduct. So suppose we have a third fixed point (z, h), together with maps of fixed points $j : (x, g) \to (z, h)$ and $k : (y, g) \to (z, h)$. Since $(x \oplus y, \varphi^{\delta}_{x,y} \circ (f \oplus g))$ has as its first component the biproduct $x \oplus y$, we already know that there exists a unique morphism $[j, k] : x \oplus y \to z$ in C (the copairing of j and k as morphisms in C) such that

$$[j, k] \circ i_x = j \quad \text{and} \quad [j, k] \circ i_y = k.$$

We show that this map is also a map of fixed points $(x \oplus y, \varphi^{\delta}_{x,y} \circ (f \oplus g)) \to (z, h)$. For this we argue that the two outer paths through the following diagram are equal

(155)
$$\begin{array}{ccc} x \oplus y & \xrightarrow{[j,l]} & z \\ {\scriptstyle f \oplus g}\downarrow & & \\ \delta x \oplus \delta y & & \downarrow{\scriptstyle h} \\ {\scriptstyle \simeq}\downarrow & \overset{\langle \delta j, \delta k \rangle}{\nwarrow} & \\ \delta(x \oplus y) & \xleftarrow[\delta[j,k]]{} & \delta z \end{array}$$

Indeed, the triangular sub-diagram commutes, because the isomorphism $\delta(x \oplus y) \to \delta x \oplus \delta y$ is equal to the pairing $\langle \delta i_x, \delta i_y \rangle$, and in the following commutative diagram

(156)
$$\begin{array}{ccccc} & & \delta z & & \\ & \overset{\delta j}{\swarrow} & \downarrow{\scriptstyle \delta[j,k]} & \overset{\delta k}{\searrow} & \\ \delta x & \xleftarrow{\delta i_x} & \delta(x \oplus y) & \xrightarrow{\delta i_y} & \delta y \\ & \underset{i_{\delta x}}{\searrow} & \downarrow{\scriptstyle \langle \delta i_x, \delta i_y \rangle} & \underset{i_{\delta y}}{\swarrow} & \\ & & \delta x \oplus \delta y & & \end{array}$$

the composition of the vertical two morphisms is equal to $\langle \delta j, \delta k \rangle$ by the uniqueness of the universal map into $\delta x \oplus \delta y$ viewed as a product. Finally, the upper sub-diagram in (155) commutes if and only if

$$\delta j \circ h \circ j = f \quad \text{and} \delta k \circ h \circ k = g$$

and this is the case, since j and k are maps of fixed points. □

NOTATION 4.1.5. Given fixed points (x, f) and (y, g) of an additive category C with duality, we denote their coproduct by

$$(x, f) \perp (y, g), \tag{157}$$

and call this operation **orthogonal sum**. Also, we sometimes call morphisms of fixed points **isometries**. This terminology is inspired by the basic motivating examples where $\mathsf{C} = \mathsf{vect}_{\mathbf{k}}$

EXAMPLE 4.1.6. Let $\mathsf{C} = \mathsf{vect}_{\mathbf{k}}$ be the category of finite dimensional vectors spaces over $\mathbf{k}$. Let $\delta V = V^*$ be the standard duality functor (which, we have seen, is additive), and let $\eta \Rightarrow \delta^\circ\delta$ be the standard natural isomorphism.

As discussed in Example 3.1.4, a fixed point for of this duality involution corresponds to a vector space V equipped with a symmetric bilinear form $B : V \times V \to \mathbf{k}$ (and strong fixed points are those for which B is non-degenerate). Morphisms of fixed points correspond to isometries.

Given two fixed points (V, B) and (V', B'), their coproduct is $(V \oplus V', B \oplus B')$, where

$$B \oplus B' : ((v, v'), (w, w')) \longmapsto B(v, w) + B(v', w'). \tag{158}$$

This bilinear form is clearly again symmetric when B and B' are (and it is non-degenerate when B and B' are). Moreover, the subspaces $V \oplus 0 \subseteq V \oplus V'$ and $0 \oplus V' \subseteq V \oplus V'$ are orthogonal with respect to $B \oplus B'$.

PROPOSITION 4.1.7. *Let $(F, \psi) : (\mathsf{C}, \delta, \eta) \longrightarrow (\mathsf{C}', \delta', \eta')$ be an additive equivariant functor between additive categories with duality. Then the induced functor $\hat{F} : \mathsf{C}^\delta \to \mathsf{C}'^{\delta'}$ defined in Proposition 3.1.6 preserves all finite coproducts.*

PROOF. Recall that $\hat{F}$ acts on objects by

$$(x, b) \longmapsto (Fx, (\psi_x)^\circ \circ Fb)$$

and on morphisms by

$$f : x \to x' \;\; \longmapsto \;\; Ff : Fx \to Fx'.$$

We start by showing that $\hat{F}$ preserves initial objects. Let (x, f) be initial in C^δ. We saw in Proposition 4.1.4 that also $(z, 0)$ is also initial in C^δ, so we have a canonical isomorphism $(x, f) \simeq (z, 0)$. This implies that $F(x, f) \simeq F(z, 0)$ in $\mathsf{C}'^{\delta'}$. But $F(z, 0) = (Fz, (\psi_z)^\circ \circ F0)$ is initial, because Fz is a zero object in C', so $F(x, f)$ must also be initial.

To prove that $\hat{F}$ preserves binary coproducts, we show that the diagram

$$\begin{array}{ccccc} & & \hat{F}((x, f) \perp (y, g)) & & \\ & \nearrow^{F(i_{(x,f)})} & \downarrow \varphi^F_{x,y} & \nwarrow^{F(i_{(y,g)})} & \\ \hat{F}(x, f) & \xrightarrow{i_{F(x,f)}} & \hat{F}(x, f) \perp \hat{F}(y, g) & \xleftarrow[i_{F(y,g)}]{} & \hat{F}(y, g) \end{array} \tag{159}$$

commutes.

First, we check that $\varphi^F_{x,y}$ is in fact a morphism

$$\hat{F}((x, f) \perp (y, g)) \longrightarrow \hat{F}(x, f) \perp \hat{F}(y, g).$$

For this we show that the diagram

$$
\tag{160}
\begin{array}{ccc}
F(x\oplus y) & \xrightarrow{\varphi^F_{x,y}} & F(x)\oplus F(y) \\
\downarrow{\scriptstyle F(f\oplus g)} & & \downarrow{\scriptstyle F(f)\oplus F(g)} \\
F(\delta^\circ x^\circ \oplus \delta^\circ y^\circ) & \xrightarrow{\varphi^F_{\delta^\circ x^\circ,\delta^\circ y^\circ}} & F\delta^\circ x^\circ \oplus F\delta^\circ y^\circ \\
\downarrow{\scriptstyle F((\varphi^\delta_{x,y})^\circ)} & \nearrow{\scriptstyle \varphi^{F\delta^\circ}_{x^\circ,y^\circ}} & \downarrow{\scriptstyle (\psi_x)^\circ\oplus(\psi_y)^\circ} \\
F\delta^\circ((x\oplus y)^\circ) & & \delta'^\circ F^\circ x^\circ \oplus \delta'^\circ F^\circ y^\circ \\
\downarrow{\scriptstyle (\psi_{x\oplus y})^\circ} & \nearrow{\scriptstyle \varphi^{\delta'^\circ F^\circ}_{x^\circ,y^\circ}} & \downarrow{\scriptstyle (\varphi^{\delta'^\circ}_{F^\circ x^\circ,F^\circ y^\circ})} \\
\delta'^\circ F^\circ((x\oplus y)^\circ) & \xleftarrow[\delta'^\circ(\varphi^F_{x,y})^\circ]{} & \delta'^\circ(F^\circ x^\circ \oplus F^\circ y^\circ)
\end{array}
$$

commutes (the left and right vertical sides of the diagram are the fixed points $\hat{F}((x,f)\perp(y,g))$ and $\hat{F}(x,f)\perp\hat{F}(y,g)$, respectively). To see this, let us start at the top of the diagram and work our way downwards.

First, the upper, square subdiagram commutes because φ^F is a binatural transformation in the pair of variables (x,y). Second, the upper, triangular subdiagram commutes because, by Lemma 1.5.12,

$$
\tag{161} F((\varphi^\delta_{x,y})^\circ) = F((\varphi^{\delta^\circ}_{x^\circ,y^\circ})^{-1}),
$$

and by (61),

$$
\tag{162} \varphi^F_{\delta^\circ x^\circ,\delta^\circ y^\circ} \circ F(\varphi^{\delta^\circ}_{x^\circ,y^\circ}) = \varphi^{F\delta^\circ}_{x^\circ,y^\circ}.
$$

Third, the middle, parallelogram-shaped subdiagram commutes by the additivity property for the natural transformation φ^F, i.e. by the general rule (60). Fourth, and last, the bottom, triangular subdiagram commutes since, by Lemma 1.5.12,

$$
\tag{163} \varphi^{\delta'^\circ F^\circ}_{x^\circ,y^\circ} = ((\varphi^{\delta' F}_{x,y})^\circ)^{-1},
$$

and by (61),

$$
\tag{164} \varphi^{\delta' F}_{x,y} = \varphi^{\delta'}_{Fx,Fy} \circ \delta'\varphi^F_{x,y}.
$$

Now we turn to showing that (159) commutes (we show the commutativity of the left-hand triangle; the other case is completely analogous). We know that the three maps involved are indeed morphisms of fixed points. Furthermore, by the definition of a morphism of fixed points, two such morphisms are equal if and only if they are equal as morphisms in the underlying category C. Thus we only need to show that

$$
\tag{165} \varphi^F_{x,y} \circ F(i_{(x,f)}) = i_{F(x,f)}
$$

as morphisms in C. But recall that $i_{(x,f)} = i_x$ and $i_{F(x,f)} = i_{Fx}$, by definition. So (165) is equivalent to

$$
\tag{166} \varphi^F_{x,y} \circ F(i_x) = i_{Fx}
$$

and this latter equation holds true by the definition of $\varphi^F_{x,y}$.

$\square$

4.2. Hyperbolization

Fix an additive category with duality $(\mathsf{C}, \delta, \eta)$. Given an object $x \in C$, there is a way to define on $x \oplus \delta^\circ x^\circ$ a "canonical" fixed point structure. It is called "hyberbolic", a name which is explained by Example 4.2.4 below.

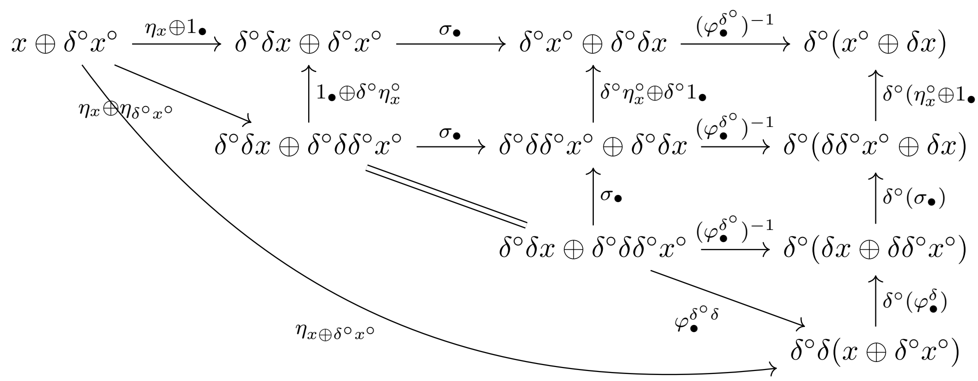
Definition 4.2.1. *Given an object x of the additive category with duality $(\mathsf{C}, \delta, \eta)$, the **hyperbolic fixed point structure** on $x \oplus \delta^\circ x^\circ$ is the morphism h_x defined by the composition*

$$x \oplus \delta^\circ x^\circ \xrightarrow{\eta_x \oplus 1_\bullet} \delta^\circ \delta x \oplus \delta^\circ x^\circ \xrightarrow{\sigma_\bullet} \delta^\circ x^\circ \oplus \delta^\circ \delta x \xrightarrow{(\varphi_\bullet^{\delta^\circ})^{-1}} \delta^\circ(x^\circ \oplus \delta x). \tag{167}$$

Proposition 4.2.2. *The pair $(x \oplus \delta^\circ x^\circ, h_x)$ is a fixed point.*

Proof. It is straightforward to see that the followng diagram commutes:

(168)



The upper horizontal composition is h_x. To see that the right-hand vertical composite is equal to $\delta^\circ(h_x^\circ)$, note that

$$(\eta_x^\circ \oplus 1_{\delta x}) = (\eta_x \oplus 1_{\delta^\circ x^\circ})^\circ$$

and

$$\varphi^\delta_{x \oplus \delta^\circ x^\circ} = (\varphi^\delta_{x \oplus \delta^\circ x^\circ})^{\circ\circ} = ((\varphi^{\delta^\circ}_{x^\circ \oplus \delta x})^{-1})^\circ.$$

□

Remark 4.2.3. When the duality involution is strong, then the hyperbolic fixed point stucture defines a strong fixed point.

Example 4.2.4. Consider the case where C is the additive category of finite dimensional $\mathbf{k}$-vector spaces and δ is the usual duality. Let the unit η be the standard embedding into the double-dual (the "symmetric case"), or the negative of that embedding (the "symplectic case"). Given a vector space $V \in \mathsf{C}$, its hyperbolization is what is known as a hyperbolic space. Namely, the hyperbolic fixed point structure on $V \oplus V^*$ corresponds to the bilinear form

$$V \oplus V^* \times V \oplus V^* \longrightarrow \mathbf{k}, \ (v, \xi, w, \zeta) \longmapsto \eta_v(\zeta) + \xi(w). \tag{169}$$

In the symmetric case this is

$$(v, \xi, w, \zeta) \longmapsto \zeta(v) + \xi(w)$$

and in the symplectic case this is

$$(v, \xi, w, \zeta) \longmapsto -\zeta(v) + \xi(w).$$

If we identify V^{**} with V, then the "coordinate matrices" of the hyperbolic fixed point structure, in the symmetric and symplectic cases respectively, are

$$\begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{bmatrix}. \tag{170}$$

Definition 4.2.5. *Let $\mathsf{C}^\times$ be the underlying groupoid of C. The **hypberbolization functor** $H : \mathsf{C}^\times \longrightarrow \mathsf{C}^\delta$ is defined as follows. Given an object $x \in \mathsf{C}$, set*

$$H(x) = (x \oplus \delta^\circ x^\circ, h_x)$$

and given a morphism $f : x \to y$ in C, set

$$H(f) = f \oplus (\delta^\circ f^\circ)^{-1}.$$

Lemma 4.2.6. *Hyperbolization is a functor $\mathsf{C}^\times \longrightarrow \mathsf{C}^\delta$.*

Proof. That H is compatible with composition and identity morphisms is easy to see. We show that, given $f : x \to y$, the morphism $f \oplus (\delta^\circ f^\circ)^{-1}$ in C is indeed a morphism of fixed points $(x \oplus \delta^\circ x^\circ, h_x) \to (y \oplus \delta^\circ y^\circ, h_y)$. Indeed, the diagram

$$\begin{array}{ccccccc}
x \oplus \delta^\circ x^\circ & \xrightarrow{\eta_x \oplus 1_\bullet} & \delta^\circ\delta \oplus \delta^\circ x^\circ & \xrightarrow{\sigma_\bullet} & \delta^\circ x^\circ \oplus \delta^\circ \delta x & \xrightarrow{(\varphi_\bullet^{\delta^\circ})^{-1}} & \delta^\circ(x^\circ \oplus \delta x) \\
\downarrow{\scriptstyle f \oplus (\delta^\circ f^\circ)^{-1}} & & \downarrow{\scriptstyle \delta^\circ\delta f \oplus (\delta^\circ f^\circ)^{-1}} & & \downarrow{\scriptstyle (\delta^\circ f^\circ)^{-1} \oplus \delta^\circ\delta f} & & \downarrow{\scriptstyle \delta^\circ((f^\circ)^{-1} \oplus \delta f)} \\
y \oplus \delta^\circ y^\circ & \xrightarrow{\eta_y \oplus 1_\bullet} & \delta^\circ\delta \oplus \delta^\circ y^\circ & \xrightarrow{\sigma_\bullet} & \delta^\circ y^\circ \oplus \delta^\circ \delta y & \xrightarrow{(\varphi_\bullet^{\delta^\circ})^{-1}} & \delta^\circ(y^\circ \oplus \delta y)
\end{array} \tag{171}$$

is commutative, and the right-hand side morphism in the diagram is

$$\delta^\circ(f \oplus (\delta^\circ f^\circ)^{-1})^\circ = \delta^\circ(f^\circ \oplus (\delta f)^{-1}) = \delta^\circ((f^\circ)^{-1} \oplus \delta f). \tag{172}$$

□

Lemma 4.2.7. *For any $x, y \in \mathsf{C}^\times$,*

$$H(x \oplus y) \simeq H(x) \perp H(y) \tag{173}$$

via the isomorphism

$$x \oplus y \oplus \delta(x \oplus y) \stackrel{1_x \oplus 1_y \oplus \varphi^\delta}{\longrightarrow} x \oplus y \oplus \delta x \oplus \delta y \stackrel{1_x \oplus \sigma \oplus 1_y}{\longrightarrow} x \oplus \delta x \oplus y \oplus \delta y. \tag{174}$$

Proof. The proof is exhibited by the following large but straightforward commutative diagram, of which we give simply a sketch (all morphisms are the obvious choices). The left-hand vertical composite is the fixed point structure of $H(x \oplus y)$ while the right-hand vertical composite is the fixed point structure of $H(x) \perp H(y)$.

$$\begin{array}{ccccc}
x \oplus y \oplus \delta(x \oplus y) & \xrightarrow{1_x \oplus 1_y \oplus \varphi^\delta} & x \oplus y \oplus \delta x \oplus \delta y & \xrightarrow{1_x \oplus \sigma \oplus 1_y} & x \oplus \delta x \oplus y \oplus \delta y \\
\downarrow{\scriptstyle \eta_{x \oplus y} \oplus 1_{\delta(x \oplus y)}} & & \downarrow & & \downarrow \\
\delta\delta(x \oplus y) \oplus \delta(x \oplus y) & \longrightarrow & \cdots & \longrightarrow & \delta\delta x \oplus \delta x \oplus \delta\delta y \oplus \delta y \\
\downarrow & & \downarrow & & \downarrow \\
\delta(x \oplus y) \oplus \delta\delta(x \oplus y) & \longrightarrow & \cdots & \longrightarrow & \delta x \oplus \delta\delta x \oplus \delta y \oplus \delta\delta y \\
\downarrow & & \downarrow & & \downarrow \\
\delta(x \oplus y \oplus \delta(x \oplus y)) & \longrightarrow & \cdots & \longrightarrow & \delta(x \oplus \delta x) \oplus \delta(y \oplus \delta y) \\
\| & & \downarrow & & \downarrow \\
\delta(x \oplus y \oplus \delta(x \oplus y)) & \longrightarrow & \cdots & \longrightarrow & \delta(x \oplus \delta x \oplus y \oplus \delta y)
\end{array} \tag{175}$$

□

In the case when C carries a strong duality involution (δ, η), we may think of the hyperbolization construction as one which combines the duality involution δ on C, restricted to $\mathsf{C}^\times$, and the duality involution on $\mathsf{C}^\times$ of "taking the inverse", in such a way as to give an *involution* on $\mathsf{C}^\times$. (If (δ, η) is not strong, the duality involution does not technically restrict to $\mathsf{C}^\times$, since then components of η may not live in $\mathsf{C}^\times$.) In the following we discuss briefly how hyperbolization may be viewed as a construction for involutions on categories.

Definition 4.2.8. *An* ***involution*** *on a category C is a functor $\tau : \mathsf{C} \to \mathsf{C}$ together with a natural transformation $\eta : 1_{\mathsf{C}} \Rightarrow \tau\tau$ such that, for all objects $x \in \mathsf{C}$,*

$$\begin{array}{ccc} \tau x & \xrightarrow{\eta_{\tau x}} & \tau\tau\tau x \\ & \searrow^{1_{\tau x}} & \uparrow{\scriptstyle \tau\eta_x} \\ & & \tau x \end{array} \tag{176}$$

We call (C, τ, η) a ***category with involution****. If C and τ are additive, then we speak of an* ***additive involution*** *and an* ***additive category with involution****.*

Remark 4.2.9. If, in the above definition, η is invertible, then $(\tau, \tau, \eta, \eta^{-1})$ is an adjunction. However, for our definition, we do not require η to be invertible (in [**Jac12**] and [**FH16**] this is required). In general, τ need not be part of an adjunction, in contrast to the case of duality involutions.

Example 4.2.10. Let $(\mathsf{C}, \delta, \eta)$ be a category with a strong duality involution. Then $\mathsf{C}^\times$ is a category with involution (τ, η), where

$$\tau(x) := \delta^\circ x^\circ \qquad x \in \mathrm{Ob}(\mathsf{C}^\times) \tag{177}$$

and

$$\tau(f) := (\delta^\circ f^\circ)^{-1} \qquad f \in \mathrm{Mor}(\mathsf{C}^\times). \tag{178}$$

Definition 4.2.11. *Let (C, τ, η) be a category with involution. A* ***fixed point*** *is a pair (x, j), where $j : x \to \tau x$ is a morphism in C such that*

$$\begin{array}{ccc} x & \xrightarrow{j} & \tau x \\ & \searrow^{\eta_x} & \downarrow{\scriptstyle \tau j} \\ & & \tau\tau x \end{array} \tag{179}$$

commutes. A ***morphism of fixed points*** *$f : (x, j) \to (x', j')$ is a morphism $f : x \to x'$ in C such that*

$$\begin{array}{ccc} x & \xrightarrow{f} & x' \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j'} \\ \tau x & \xrightarrow{\tau f} & \tau x' \end{array} \tag{180}$$

commutes. Fixed points and their morphisms assemble to a category, which we denote by C^τ.

Definition 4.2.12. *Let (C, τ, η) be an additive category with involution. Given an object $x \in \mathsf{C}$, we define the* ***hyperbolic fixed point structure*** *on $x \oplus \tau x$ to be the morphism given by the composite*

$$x \oplus \tau x \xrightarrow{\eta_x \oplus 1_{\tau x}} \tau\tau x \oplus \tau x \xrightarrow{\sigma_\bullet} \tau x \oplus \tau\tau x \xrightarrow{\varphi_\bullet^{-1}} \tau(x \oplus \tau x), \tag{181}$$

where $\sigma_\bullet$ is the symmetry isomorphism of C, and $\varphi_\bullet$ is the coherence isomorphism for the additivity of τ (compare with Definition 4.2.1).

LEMMA 4.2.13. *(C, τ, η) be an additive category with involution. For any x, the hyperbolic fixed point structure (181) on $x \oplus \tau x$ defines a fixed point.*

PROOF. The proof is completely analogous to the proof of Proposition 4.2.2. □

LEMMA 4.2.14. *Let (C, τ, η) be a category with involution. Then **hyperbolization** is the functor $H : \mathsf{C} \longrightarrow \mathsf{C}^\tau$ defined by*

$$H(x) = x \oplus \tau x, \qquad x \in Ob(\mathsf{C}),$$

and

$$H(f) = f \oplus \tau f, \qquad f \in Mor(\mathsf{C}).$$

PROOF. The proof is completely analogous to the proof of Lemma 4.2.6. □

REMARK 4.2.15. Definition 4.2.5 is a special case of Lemma 4.2.14, for τ as in Example 4.2.10.

In Example 4.2.10, two duality involutions on $\mathsf{C}^\times$ are composed to give an involution τ. We describe this construction more generally.

LEMMA 4.2.16. *Let C be a category equipped with duality involutions (δ_1, η_1) and (δ_2, η_2). Let $\tau := \delta_2^\circ \delta_1$ and let $\eta := \eta_2 \star \eta_1$. Suppose that $\delta_2^\circ \delta_1 = \delta_1^\circ \delta_2$, and that $\tau \star \eta_1 = \eta_1 \star \tau$ and $\tau \star \eta_2 = \eta_2 \star \tau$. Then (τ, η) defines an involution on C.*

PROOF. Note that, taking opposites, also $\delta_2 \delta_1^\circ = \delta_1 \delta_2^\circ$ holds, and so $\eta : 1_{\mathsf{C}} \Rightarrow \delta_2^\circ \delta_2 \delta_1^\circ \delta_1 = \delta_2^\circ \delta_1 \delta_2^\circ \delta_1 = \tau\tau$. For (τ, η) to be an involution, we need that $\tau \star \eta = \eta \star \tau$. This clearly follows from the hypotheses; indeed

$$\tau \star \eta = \tau \star \eta_2 \star \eta_1 = \eta_2 \star \tau \star \eta_1 = \eta_2 \star \eta_1 \star \tau = \eta \star \tau. \tag{182}$$

□

EXAMPLE 4.2.17. Let C be the additive category of linear relations, with objects finite-dimensional **k**-vector spaces. Let (δ_1, η_1) be the duality involution on C discussed in Example 2.2.9, i.e. given a linear relation $R \subseteq V \oplus W$,

$$\delta_1 R = R^* = \{(\chi, \xi) \in W^* \oplus V^* \mid \chi(w) = \xi(v) \ \forall (v, w) \in R\}, \tag{183}$$

and let (δ_2, η_2) be the duality involution on C discussed in Example 2.2.5, i.e.

$$\delta_2 R = R^\dagger = \{(w, v) \in W \oplus V \mid (v, w) \in R\}. \tag{184}$$

Note that δ_1 generalizes the operation of taking the adjoint of a linear map, and δ_2 generalizes the operation of taking the inverse of an invertible linear map.

We check that $\tau = \delta_2^\circ \delta_1$ and $\eta = \eta_1 \star \eta_2$ satisfy the hypothesis of the previous Lemma. It is straightforward to see that $\delta_2^\circ \delta_1 = \delta_1^\circ \delta_2$. Indeed, on objects we have that $\delta_2^\circ \delta_1 V = V^* = \delta_1^\circ \delta_2 V$, and for any linear relation $R \subseteq V \oplus W$, and it is easy to see that $(R^*)^\dagger = (R^\dagger)^*$. Furthermore, the components of η_2 are identity morphisms, so $\tau \star \eta_2 = \eta_2 \star \tau$ holds trivially. For η_1, the components are the natural isomorphisms $V \to V^{**}$, and we have, for every object V,

$$\tau(\eta_{2,V}) = (\eta_{2,V}^*)^\dagger = \{(L \circ \eta_{2,V}, L) \in V^* \oplus V^{***} \mid L \in V^{***}\}, \tag{185}$$

and

$$\eta_{2,\tau V} = \eta_{2,V^*} = \{(\xi, ev_\xi) \in V^* \oplus V^{***} \mid \xi \in V^*\}. \tag{186}$$

On the one hand, given $(\xi, ev_\xi) \in \eta_{2,\tau V}$, we have that $ev_\xi \circ \eta_{2,V} = \xi$, so $(\xi, ev_\xi) \in \tau(\eta_{2,V})$. On the other hand, given $(L \circ \eta_{2,V}, L) \in \tau(\eta_{2,V})$, since η_{2,V^*} is surjective (we are in finite dimensions), there exists $\xi \in V^*$ such that $ev_\xi = L$, and thus, since $ev_\xi \circ \eta_{2,V} = \xi$, we find that $(L \circ \eta_{2,V}, L) \in \eta_{2,\tau V}$. △

4.3. Krull-Schmidt

Here we discuss versions of the "Krull-Schmidt theorem" for additive categories and for categories of fixed points coming from additive categories with duality. The material is based on [**QSS79**]; see also the more detailed exposition in [**Knu91**].

Let C be a category with all finite coproducts, and let $x \in \text{ob}(C)$. By a **decomposition** of x we mean an isomorphism

$$x \simeq \coprod_{i=1}^{n} x_i \tag{187}$$

for some objects $x_1, ..., x_n \in \text{ob}(C)$ and some $n \in \mathbb{N}$. The objects x_i are called **summands** of x. An object x is **indecomposable** if $x \simeq y \amalg z$ implies that either y or z is an initial object. In other words, x is indecomposable if it does not admit any non-trivial decomposition.

We say "**the Krull-Schmidt theorem holds in** C" if

(1) Any object $x \in \mathsf{C}$ has a decomposition $x \simeq \coprod_{i=1}^{n} x_i$ whose direct summands x_i are indecomposable;
(2) If $x \simeq \coprod_{i=1}^{m} y_i$ is another decomposition into indecomposables, then $m = n$ and there is some permutation π of $\{1, ..., n\}$ such that $y_i \simeq x_{\pi(i)}$ for all $i = 1, .., n$.

This means, in other words, that every object has a decomposition into indecomposables, and such a decomposition is "essentially unique".

If $x \simeq \coprod_{i=1}^{n} x_i$ is a decomposition into indecomposables, we say that x is of the **type** $\{t_1, ..., t_k\}$ if each x_i is isomorphic to some t_j in $\{t_1, ..., t_k\}$. The idea is that a "type" for x captures the set of isomorphism classes of summands of x. We do not, a priori, assume that the t_j are pairwise non-isomorphic, so there may be some redundancy in terms of isomorphism classes; we do however assume that for every t_j in $\{t_1, ..., t_k\}$ there is a summand of x which is isomorphic to t_j. For example, if $x_i \simeq t \; \forall i = 1, .., n$ for some t, then we can say that x is of type $\{t\}$, but we cannot say that x is of type $\{t, t'\}$ unless $t \simeq t'$.

4.3.1. Additive categories. Assume now that C is an additive category. We give conditions under which the Krull-Schmidt theorem holds in C.

Definition 4.3.1. *We label the following two properties that an additive category C may have:*

P1 *All idempotents of C split.*
P2 *For every object $x \in \mathsf{C}$ there exists a decomposition $x \simeq \oplus_{i=1}^{n} x_i$ into indecomposables such that, for each summand, $End(x_i)$ is a local ring.*

If the first property holds, this means that all (conjugate pairs of) idempotents correspond to (binary) decompositions. If the second property holds, we are guaranteed to always have decompositions into indecomposables, and the endomorphism rings of indecomposables are "well-behaved". The following is Theorem 5.2.1 in [**Knu91**].

THEOREM 4.3.2. *If C satisfies **P1** and **P2**, then the Krull-Schmidt theorem holds in C.*

DEFINITION 4.3.3. *We call an additive category a **Krull-Schmidt category** if **P1** and **P2** hold.*

Precise conditions under which an additive category is Krull-Schmidt are given, for example, in [**Kra15**].

4.4. Orthogonal decompositions

Let $(\mathsf{C}, \delta, \eta)$ be an additive category with duality. In this section we assume that C is a Krull-Schmidt category, and that C is idempotent complete (see Definition 1.5.23).

From Proposition 4.1.4 we know that the category C_s^δ of strong fixed points has finite coproducts. We wish to formulate conditions on C under which statements similar to the Krull-Schmidt theorem hold in C_s^δ.

The following expresses a certain regularity property for endomorphism rings of an additive category. Given any ideal I of a ring E, there is a notion of I-adic topology and I-adic completeness, see [**Knu91**] Section II.4.5. In particular, one may consider the ideal given by the radical $\mathrm{Rad}(E)$ of E.

DEFINITION 4.4.1.

P3 *For every object $x \in \mathsf{C}$, the ring $\mathrm{End}(x)$ is Rad$(\mathrm{End}(x))$-adically complete.*

The purpose of **P3** is that it guarantees that one may lift idempotents from $\mathrm{End}(x)/\mathrm{Rad}(\mathrm{End}(x))$ to $\mathrm{End}(x)$, see [**Knu91**], II.4.5.4.

REMARK 4.4.2. Let E be a ring. If E is artinian, then E is $\mathrm{Rad}(\mathrm{End}(x))$-adically complete (see [**Knu91**], Section II.4.5). The rings that we consider in Part 2 of this thesis are subrings of endomorphism algebras of finite-dimensional vector spaces, and hence they are in particular artinian.

The following is Theorem II.6.3.1 in [**Knu91**].

THEOREM 4.4.3. *Let C be an additive category in which the Krull-Schmidt theorem holds. Let $(x, f) \in \mathsf{C}_s^\delta$ be a strong fixed point, where $x \in ob(C)$ is of type $\{t_1, ..., t_k\}$, with $t_i \not\simeq t_j$ for $i \neq j$. Then there exits a decomposition*

$$(x, f) \simeq (x_1, f_1) \perp \cdots \perp (x_n, f_n), \tag{188}$$

*such that each (x_i, f_i) is of type $\{t_i, t_i^*\}$. If, in addition, C satisfies property **P3**, then this decomposition is unique up to isometries and permutations of the summands.*

REMARK 4.4.4. The decomposition in the previous theorem is one into "isotypic components"; the individual summands may themselves be decomposable. In general, we do not have a full Krull-Schmidt theorem for C_s^δ. An orthogonal decomposition into indecomposable summands does always exist, but the uniqueness part of Krull-Schmidt may fail. We illustrate this in Example 4.4.5 below.

EXAMPLE 4.4.5. Let $\mathbf{k} = \mathbb{Z}/5\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}\}$. Let C be the additive category of finite-dimensional $\mathbf{k}$-vector spaces, equipped with the standard duality involution (the duality functor sends a vector space V to its dual $V^* = \mathrm{Hom}(V, \mathbf{k})$, and the components of the unit for the duality involution are the standard embeddings $V \to V^{**}$). Strong fixed points are thus pairs (V, B) where $V \in \mathsf{C}$ and $B : V \to V^*$ is a non-degenerate symmetric bilinear form.

We will now give an example of a strong fixed point (V, B) which has two different orthogonal decompositions into indecomposable fixed points such that the respective summands are *not* isomorphic (irrespective of ordering). Let $V = \mathbf{k}^2$, and consider the following symmetric bilinear forms on $\mathbf{k}^2$, given in terms of their coordinate matrices (with respect to the standard basis on $\mathbf{k}^2$):

$$B_1 := \begin{bmatrix} \overline{3} & \overline{0} \\ \overline{0} & \overline{3} \end{bmatrix} \qquad B_2 := \begin{bmatrix} \overline{1} & \overline{0} \\ \overline{0} & \overline{4} \end{bmatrix}. \tag{189}$$

Note that

$$(\mathbf{k}^2, B_1) = (\mathbf{k}, [\ \overline{3}\ ]) \perp (\mathbf{k}, [\ \overline{3}\ ]) \tag{190}$$

and

$$(\mathbf{k}^2, B_2) = (\mathbf{k}, [\ \overline{1}\ ]) \perp (\mathbf{k}, [\ \overline{4}\ ]) \tag{191}$$

are orthogonal decompositions into indecomposable summands, and the type (an isomorphism class in C) of each of the four summands is given by $\mathbf{k} \in \mathsf{C}$.

We claim that $(\mathbf{k}^2, B_1)$ and $(\mathbf{k}^2, B_2)$ are isometric (i.e. they are isomorphic as strong fixed points), but that $(\mathbf{k}, [\ \overline{3}\ ])$ is neither isometric to $(\mathbf{k}, [\ \overline{1}\ ])$ nor to $(\mathbf{k}, [\ \overline{4}\ ])$.

An isometry from $(\mathbf{k}^2, B_2)$ to $(\mathbf{k}^2, B_1)$ is given explicitly in coordinates by the matrix

$$\begin{bmatrix} \overline{1} & \overline{2} \\ \overline{1} & \overline{3} \end{bmatrix}. \tag{192}$$

Indeed,

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 15 \\ 15 & 39 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \pmod 5. \tag{193}$$

Now we observe that for any non-zero elements $x, y \in \mathbf{k}$, there is an isometry between $(\mathbf{k}, [x])$ and $(\mathbf{k}, [y])$ if and only if x and y represent the same class in the square class group $\mathbf{k}^\times/(\mathbf{k}^\times)^2$. Indeed, an isometry is given by a 1×1 matrix $[s]$ such that $[s]\,[x]\,[s] = [s^2 x] = [y]$. In $\mathbf{k} = \mathbb{Z}/5\mathbb{Z}$ the non-zero squares are $\{\overline{1}, \overline{4}\}$ so the two equivalence classes in $\mathbf{k}^\times/(\mathbf{k}^\times)^2$ are $\{\overline{1}, \overline{4}\}$ and $\{\overline{2}, \overline{3}\}$. In particular it follows that $(\mathbf{k}, [\ \overline{3}\ ])$ is not isometric to $(\mathbf{k}, [\ \overline{1}\ ])$ nor to $(\mathbf{k}, [\ \overline{4}\ ])$. △

We now turn to studying indecomposable strong fixed points. The key result is Proposition 4.4.16 below, which leads to a general strategy for classifying indecomposable strong fixed points. This strategy is sketched briefly in a remark at the end of the section, and applied in detail, in the next part of the thesis, to several example cases.

LEMMA 4.4.6. *Let (x, b) be a strong fixed point. Suppose $x = u \oplus w$, with $u \perp w$. Then, by restriction, b induces a strong fixed point structure on u and w, and*

$$(x, b) \simeq (u, b|_u) \perp (w, b|_w).$$

PROOF. Consider the map $u \oplus w \xrightarrow{b} \delta(u \oplus w) \xrightarrow{(\varphi^\delta)^{-1}} \delta u \oplus \delta w$, and let its "coordinate matrix" be

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \tag{194}$$

That means, for instance, that the map a_{12} is the composite

$$w \xrightarrow{i_w} x \xrightarrow{b} \delta x \xrightarrow{(\varphi^\delta)^{-1}} \delta u \oplus \delta w \xrightarrow{p_{\delta u}} \delta u. \tag{195}$$

We claim that in fact, the maps a_{12} and a_{21} are the zero morphism. Indeed, consider a_{12}, i.e. (195). Since δx is a biproduct of δu and δw via the maps δp_u, δp_w, δi_u, and δi_w, and since φ^δ is the map guaranteed by the universal property of the biproducts δx and $\delta u \oplus \delta w$ viewed as products, we have

$$\delta x \xrightarrow{(\varphi^\delta)^{-1}} \delta u \oplus \delta w \xrightarrow{p_{\delta u}} \delta u = \delta x \xrightarrow{\delta i_u} \delta u. \tag{196}$$

Thus

$$a_{12} = w \xrightarrow{i_w} x \xrightarrow{b} \delta x \xrightarrow{\delta i_u} \delta u, \tag{197}$$

which is zero by the definition of u being orthogonal to w. To show that $a_{21} = 0$, we may use the same argument with the roles of w and u exchanged.

Now it is a general fact of additive categories that a matrix of maps of the form

$$\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} \tag{198}$$

is invertible if and only if the maps a_{11} and a_{22} are (c.f. [**Knu91**], II. Lemma (1.1.6)). The overall map given by this matrix is $\varphi^\delta \circ b$; it is an isomorphism because b is a strong fixed point. Thus the maps a_{11} and a_{22} are isomorphisms. And they are precisely the restrictions of b to u and w, respectively. Finally, the matrix (203) corresponds to the map $\varphi^\delta \circ (b|_u \oplus b|_w)$ as in the definition (151) of the orthogonal sum of fixed points. □

LEMMA 4.4.7. *Let (x, b) be a strong fixed point, and let $e \in End(x)$ be an idempotent. Then $im(e)^\perp$ exists and $im(e)^\perp \simeq ker(e^t)$ as subobjects. Equivalently, $im(e^t)^\perp \simeq ker(e)$ as subobjects.*

PROOF. The two statements are equivalent, because $e^{tt} = e$. We show that $\mathrm{im}(e^t)^\perp \simeq \ker(e)$. By Lemma 3.4.6, $\ker(e) \perp \mathrm{im}(e^t)$. We need to show that $\ker(e)$ is an upper bound on all subobjects orthogonal to $\mathrm{im}(e^t)$, thus proving that $\mathrm{im}(e^t)^\perp$ exists and is isomorphic to $\ker(e)$.

Let $e^t = x \xrightarrow{p} w \xrightarrow{i} x$ be a splitting of e^t, where $w = \mathrm{im}(e^t)$, and let y be an arbitrary subobject of x which is orthogonal to $\mathrm{im}(e^t)$. This means, by definition, that

$$y \xrightarrow{i_y} x \xrightarrow{b} \delta x \xrightarrow{\delta i} \delta w = 0. \tag{199}$$

From this we find

$$0 = \delta p \circ \delta i \circ b \circ i_y = e^t \circ b \circ i_y = b \circ e \circ i_y, \tag{200}$$

which implies that also $e \circ i_y = 0$. By the definition of $\ker(e)$ it follows that $y \le \ker(e)$. □

LEMMA 4.4.8. *Let (x, b) be a strong fixed point. If $x \simeq u \oplus w$, then $u^\perp$ and $(u^\perp)^\perp$ exist, and $(u^\perp)^\perp \simeq u$ as subobjects.*

PROOF. Let e be the idempotent associated with the decomposition $x \simeq u \oplus w$, with $u = \mathrm{im}(e)$. By Lemma 4.4.7, $u^\perp = \mathrm{im}(e)^\perp \simeq \ker(e^t)$ and

$$(u^\perp)^\perp = \ker(e^t)^\perp \simeq \mathrm{im}(e) = u. \tag{201}$$

□

Lemma 4.4.9. *Let (x, b) be a strong fixed point. If $x \simeq u \oplus w$, with associated idempotents $(e, 1 - e)$, then $x \simeq u^{\perp} \oplus w^{\perp}$, with idempotents $(1 - e^t, e^t)$.*

Proof. This follows from Lemma 4.4.7: the decomposition assocated to $(1 - e^t, e^t)$ is

$$\mathrm{im}(1 - e^t) \oplus \ker(1 - e^t) = \ker(1 - e)^{\perp} \oplus \mathrm{im}(1 - e)^{\perp} = \mathrm{im}(e)^{\perp} \oplus \ker(e)^{\perp}. \tag{202}$$

□

Lemma 4.4.10. *Let (x, b) be a strong fixed point.*

1. *Decompositions $x \simeq u \oplus w$ with $u = w^{\perp}$ and $w = u^{\perp}$ correspond to pairs of conjugate idempotents $(e, 1 - e)$ such that $e = e^t$.*
2. *Decompositions $x \simeq u \oplus w$ with $u = u^{\perp}$ and $w = w^{\perp}$ correspond to pairs of conjugate idempotents $(e, 1 - e)$ such that $e = 1 - e^t$.*

Proof. We show Part (1). Part (2) may be proved analogously. Since C is idempotent complete, we have $x \simeq \mathrm{im}(e) \oplus \ker(e) = u \oplus w$.

If $e = e^t$, then $w^{\perp} \simeq \ker(e)^{\perp} \simeq \mathrm{im}(e^t) \simeq \mathrm{im}(e) \simeq u$ and $u^{\perp} \simeq \mathrm{im}(e)^{\perp} \simeq \ker(e^t) \simeq \ker(e) = w$ as subobjects. Conversely, if $\mathrm{im}(e) \simeq \ker(e)^{\perp} \simeq \mathrm{im}(e^t)$ and $\ker(e) \simeq \mathrm{im}(e)^{\perp} \simeq \ker(e^t)$, then by Lemma 1.5.13,

$$\mathrm{im}(e) \oplus \ker(e) = \mathrm{im}(e^t) \oplus \ker(e^t)$$

are equivalent decompositions of x, and so $e = e^t$ by Lemma 1.5.18. □

Lemma 4.4.11. *Let (x, b) be a strong fixed point. Suppose $x = u \oplus w$, with u and w isotropic. Then (x, b) is isomorphic to a hyperbolic fixed point, i.e.*

$$(x, b) \simeq H(u) = (u, h_h)$$

in C^{δ}_{s}.

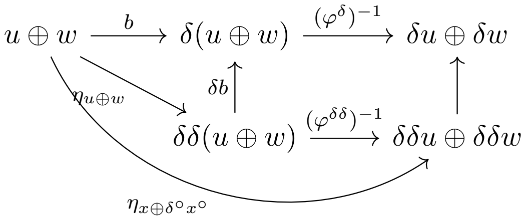
Proof. Consider the matrix

$$[b] := \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{203}$$

of the map $u \oplus w \xrightarrow{b} \delta(u \oplus w) \xrightarrow{(\varphi^{\delta})^{-1}} \delta u \oplus \delta w$. Because $u \perp u$ and $w \perp w$, we have $a_{11} = a_{22} = 0$.

Consider now the commutative diagram

(204)



The upper horizontal composite morphism is

$$\begin{bmatrix} 0 & a_{12} \\ a_{21} & 0 \end{bmatrix} \tag{205}$$

while the lower outer composite morphism is the composition

$$\begin{bmatrix} 0 & \delta a_{21} \\ \delta a_{12} & 0 \end{bmatrix} \begin{bmatrix} \eta_u & 0 \\ 0 & \eta_w \end{bmatrix}. \tag{206}$$

This shows that $a_{21} = \delta a_{12} \circ \eta_u$ and $a_{12} = \delta a_{21} \circ \eta_w$.

To finish the proof, we show that (x, b) and the hyperbolic fixed point $(w \oplus \delta w, h_w)$ are isomorphic as fixed points. We claim that the map $1_u \oplus a_{21} : u \oplus w \to u \oplus \delta u$ is such an isomorphism. To see this, recall that h_w is such that the matrix of $(\varphi^\delta)^{-1} \circ h_w$ is

$$[h] := \begin{bmatrix} 0 & 1_w \\ \eta_u & 0 \end{bmatrix}. \tag{207}$$

That $f := 1 \oplus a_{21}$ is a morphism of fixed points $(x, b) \to (w \oplus \delta w, h_w)$ now follows from the fact that, in terms of coordinate matrices, we have "$[b] = [\delta f][h][f]$":

$$\begin{bmatrix} 0 & a_{12} \\ \delta a_{21} \circ \eta_w & 0 \end{bmatrix} = \begin{bmatrix} 1_{\delta u} & 0 \\ 0 & \delta a_{12} \end{bmatrix} \begin{bmatrix} 0 & 1_{\delta w} \\ \eta_w & 0 \end{bmatrix} \begin{bmatrix} 1_u & 0 \\ 0 & a_{12} \end{bmatrix}. \tag{208}$$

□

Definition 4.4.12. *Let x be an object of the additive category C. We say that **Fitting's lemma holds for** x if for every $f \in \mathrm{End}(x)$, the kernels and images of the powers of f exist and there is an integer r such that*

$$x \simeq \mathrm{Ker}\, f^r \oplus \mathrm{Im}\, f^r. \tag{209}$$

*We say that **Fitting's lemma** holds for C is it holds for every object of C.*

Example 4.4.13. Consider the case where C is the category of finite-dimensional vector spaces over a fixed ground field. In this case Fitting's lemma holds (this is the original setting of Fitting's lemma). Indeed, given a finite-dimensional vector space V and an endomorphism $f \in \mathrm{End}(V)$, the sequence of subspaces $\mathrm{Ker}(f)$, $\mathrm{Ker}(f^2)$, $\mathrm{Ker}(f^3)$, ... is non-decreasing. Let r be the smallest integer such that $\mathrm{Ker}(f^r) = \mathrm{Ker}(f^{r+1})$; it is easily seen that then $\mathrm{Ker}(f^r) = \mathrm{Ker}(f^{r+k})$ for any integer k.

We now wish to show that $V \simeq \mathrm{Ker}\, f^r \oplus \mathrm{Im}\, f^r$. Since $\dim(\mathrm{Ker} f) + \dim(\mathrm{Im} f) = \dim V$, it is sufficient to show that $\mathrm{Ker}\, f^r \cap \mathrm{Im}\, f^r = 0$. Suppose $v \in \mathrm{Ker}\, f^r \cap \mathrm{Im}\, f^r = 0$. Then, in particular, there exists $w \in V$ such that $v \in f^r w$. But $0 = f^r v = f^r f^r w$ implies that $w \in \mathrm{Ker}\, f^{2r} = \mathrm{Ker}\, f^r$, so $v = f^r w = 0$. △

Corollary 4.4.14. *Assume Fitting's lemma holds for an indecomposable object x, and let $f \in \mathrm{End}(x)$. Then either f if nilpotent or f is both monic and epic.*

Proof. Since x is indecomposable, either $\mathrm{Ker}\, f^r = 0$ or $\mathrm{Im}\, f^r = 0$ must hold. In the first case, $\mathrm{Ker}\, f^r = 0 \Rightarrow \mathrm{Ker}\, f = 0 \Rightarrow f$ monic. Furthermore, we have $\mathrm{Im}\, f^r = x$, which implies $\mathrm{Im}\, f = x$, which implies that f is epic. In the second case, $\mathrm{Ker}\, f^r = x$ holds, which means that f is nilpotent. □

Lemma 4.4.15. *Assume that Fitting's lemma holds in C. Let (x, b) be a strong fixed point which is indecomposable in C_s^δ. If f is an endomorphism of (x, b) such that $f^t = \pm f$, then either f is nilpotent or f is both monic and epic.*

Proof. Since Fitting's lemma is assumed to hold, there exists a non-negative integer d such that $x = \mathrm{Ker} f^d \oplus \mathrm{Im} f^d$. Since $(f^d)^t = (f^t)^d = \pm f^d$, this decomposition of x is one into orthogonal summands (see Lemma 3.4.6). Because (x, b) is assumed to be an indecomposable fixed point, by Lemma 4.4.6 either $\mathrm{Ker} f^d$ or $\mathrm{Im} f^d$ must be zero. □

The following is based on Lemma 2 in [**Ser87**].

PROPOSITION 4.4.16. *Assume that Fitting's lemma holds in* C*. Let* (x, b) *be a strong fixed point which is indecomposable in* C_s^δ*, but such that* x *is decomposable in* C*. Then there exists an indecomposable* $y \in \mathsf{C}$ *such that*

$$(x, b) \simeq H(y) = (y \oplus \delta^\circ y^\circ, h_y),$$

i.e. such that (x, b) *is isomorphic to the hyperbolization of* y*.*

PROOF. Because x is decomposable in C, there exists a non-trivial idempotent $e_1 \in \mathrm{End}(x)$. After two modifications, e_1 will be conjugated into an idempotent endomorphism e satisfying the $e^t e = 0 = ee^t$ and $e + e^t = 1$. By Lemma 4.4.10 and Lemma 4.4.11 this shows that (x, b) is a hyperbolization of some y. That y must be indecomposable follows from Lemma 4.2.7, and the fact that (x, b) is assumed indecomposable.

The idempotent e_1^t is also an endomorphism of x, and $e_1^t \neq e_1$, since otherwise, by Lemma 4.4.10 and Lemma 4.4.6, (x, b) would be decomposable in C_s^δ. Set $\rho_1 = e_1 e_1^t$. Note that ρ_1 is self-adjoint and lies in $\mathrm{End}(x)$. By Lemma 4.4.15, ρ_1 must be nilpotent: ρ_1 cannot be monic/epic, since e_1 and e_1^t have nontrivial kernels and cokernels.

Now set $h_1 := s(\rho_1)$, where $s(X)$ is the binomial series for $(1 - X)^{1/2}$; $s(\rho_1)$ is well-defined because ρ_1 is nilpotent, which implies that the power series is just a polynomial in ρ_1. Note that $h_1 \in \mathrm{End}(x)$, and that h_1 is also self-adjoint. Furthermore, h_1 is invertible, its inverse being defined by substituting ρ_1 in the binomial series for $(1 - X)^{-1/2}$.

Define $e_2 := h_1 e_1 h_1^{-1}$, and note that e_2 lies in $\mathrm{End}(x)$ and is again a non-trivial idempotent. Furthermore,

$$e_2^t e_2 = h^{-1} e_1 h_1^2 e_1^t h_1^{-1} = h_1^{-1} e_1 (1 - e_1 e_1^t) e_1^t h_1^{-1} = h_1^{-1}(e_1 e_1^t - e_1 e_1^t) h_1^{-1} = 0.$$

We are half-way there. Now $\rho_2 := e_2 e_2^t$ is a nilpotent, self-adjoint element of $\mathrm{End}(x)$, and $h_2 := s(\rho_2)$ is again an invertible, self-adjoint endomorphism of x. Then $e := h_2^{-1} e_2 h_2 \in \mathrm{End}(x)$ is a non-trivial idempotent such that

$$ee^t = h_2^{-1} e_2 h_2^2 e_2^t h_2^{-1} = h_2^{-1} e_2 (1 - e_2 e_2^t) e_2^t \tilde{h}^{-1} = h_2^{-1}(e_2 e_2^t - e_2 e_2^t) h_2^{-1} = 0$$

and

$$\begin{aligned} e^t e &= h_2 e_2^t (h_2^{-2}) e_2 h_2 = h_2 e_2^t (1 - e_2 e_2^t)^{-1} e_2 h_2 \\ &= h_2 e_2^t (1 + e_2 e_2^t) e_2 h_2 = h_2 (e_2^t e_2 + e_2^t e_2 e_2^t e_2) h_2 = 0, \end{aligned}$$

since $e_2^t e_2 = 0$. Furthermore, $e + e^t \in \mathrm{End}(x)$ is idempotent: $(e + e^t)^2 = e^2 + e^t e + ee^t + (e^t)^2 = e + e^t$. But $e + e^t$ is also self-adjoint, so, by Lemma 4.4.10 and Lemma 4.4.6, $e + e^t$ must be a trivial idempotent. It cannot be that $e + e^t = 0$, since this would imply $e^t = -e$, whence $0 = e^t e = -e^2 = e$, a contradiction to $e \neq 0$. Thus $e + e^t = 1$. □

The previous Lemma shows that every indecomposable strong fixed point (x, b) is such that either x is indecomposable in C, or (x, b) is the hyperbolization of an indecomposable object $y \in \mathsf{C}$. In the former case we say that (x, b) is of **non-split** type; in the latter case we say that (x, b) is of **split** type.

REMARK 4.4.17. In the subsequent part of the thesis we will be interested in classifying indecomposable strong fixed points, given certain underlying additive categories C. If C satisfies the necessary hypotheses, and if we already have a classification of the indecomposable objects in C, then Lemma 4.4.16 essentially reduces the classification problem to that of classifying the non-split fixed points, i.e. fixed points where the underlying object in C is indecomposable. See Chapter 6, and in particular Remark 6.1.1, for a detailed illustration of how this can work.

Part 2

Classification problems in symplectic linear algebra

This part of the thesis is about symplectic linear algebra, and in particular about how various classification problems may be cast in the language of categories with duality. For those unfamiliar with symplectic geometry, I have included a short initial primer on the most basic rudiments of the subject, forming Chapter 5.

I first worked on questions of linear symplectic algebra in the context of my Master's thesis [**Lor15**], for which my advisor was Alan Weinstein. Through continuing collaboration with him, three subsequent papers followed [**LW15**] [**LW16**] [**HLW19**], each addressing classification questions in symplectic linear algebra. While working toward the third paper we recruited the help and expertise of Christian Herrmann (who vigorously joined our undertaking), and we also eventually found, in the literature, general frameworks which addressed the kinds of classification problems we had been studying. On the one hand, we found the paper [**Ser87**] V. Sergeichuk, and on the other hand the works [**QSS79**] [**Sch75**] of W. Scharlau and collaborators. We did not try to fully adopt either of these frameworks and embed our results in their language, both because our work, being already progressed, would have required a substantial reformulation, and because we decided to keep our exposition as elementary as possible. Nevertheless, both frameworks offered valuable material towards solving our specific problem.

In this thesis I have tried to create a bridge between the results of our paper [**HLW19**] on symplectic poset representations and isotropic triples, which is the content of Chapters 7 and 8 below, and the general frameworks of Sergeichuk and Scharlau et al., respectively. To this aim, I have included a minimum of necessary material on additive categories with duality (in Chapter 4 above), and used the questions of classifying linear Hamiltonian vector fields and linear symplectomorphisms as examples to illustrate the general approach. These two "case studies" are presented in Chapter 6 below, and for these I also borrow various methods and results from our paper [**HLW19**]. Thus Chapter 6 serves both as a "warm up" for the more involved analysis of isotropic triples in Chapter 8, and also serves as a guide to the connections with the broader categorical picture, since these connections are not explicitly made in the material on symplectic poset representations and isotropic triples.

CHAPTER 5

Primers and preliminaries

In this chapter, we give a quick primer on symplectic geometry, followed by a small excursion on the role of symplectic geometry in classical mechanics, for those readers unfamiliar with this topic. All vector spaces are assumed finite-dimensional over a ground field not of characteristic 2.

5.1. What is symplectic geometry?

One way to think of symplectic geometry is that it is a skew-symmetric cousin of orthogonal geometry, of which Euclidean geometry is a special case. By an **orthogonal geometry** we mean a $\mathbf{k}$-vector space V equipped with a non-degenerate, symmetric bilinear form

$$B : V \oplus V \longrightarrow \mathbf{k}.$$

Non-degenerate means that if $B(v, w) = 0\ \forall w \in V$, then $v = 0$ follows. Symmetric means $B(v, w) = B(w, v)\ \forall v, w \in V$. If the ground field is the real number field, $\mathbf{k} = \mathbb{R}$, and if B is additionally positive definite ($B(v, v) \geq 0$, with equality if and only if $v = 0$), then we speak of a **Euclidean geometry**.

A **symplectic geometry** is a $\mathbf{k}$-vector space V equipped with a bilinear form B which is non-degenerate and *skew*-symmetric, i.e. $B(v, w) = -B(w, v)\ \forall v, w \in V$. In this case, the bilinear form is called a **symplectic form** or a **symplectic structure**, and is often denoted with the letter "ω" (though we will sometimes use other letters, such as "B").

In the case of an orthogonal geometry, the bilinear form B is sometimes called a **metric structure**. This is because the associated function $V \longrightarrow \mathbf{k}, v \longmapsto B(v, v)$ is sometimes thought of as a measurement of "length". In the case of a Euclidean geometry, not only does $\|v\| := B(v, v)$ encode the length of a vector, but B also encodes angles between vectors via the well-known formula

$$\cos(\theta) = \frac{B(v, w)}{\|v\|\|w\|},$$

where $\theta \in [0, \pi]$ is the angle between v and w.

For a symplectic geometry (V, ω), there is no naive analogue of length, because the skew-symmetry of ω implies that

$$\omega(v, v) = 0\ \forall v \in V.$$

Instead of lengths and angles, symplectic geometry is about measuring signed 2-dimensional areas associated to ordered pairs of vectors. To explain this in a bit more detail, let us introduce a few facts and terms.

First of all, for any symplectic geometry (V, ω), the space V is necessarily even dimensional. And any such V, of dimension $2n$, say, admits an ordered basis

$$(e_1, ..., e_n, f_1, ..., f_n),$$

called a **symplectic basis**[1], such that

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0 \ \forall i, j \quad \text{and} \quad \omega(e_i, f_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else.} \end{cases}$$

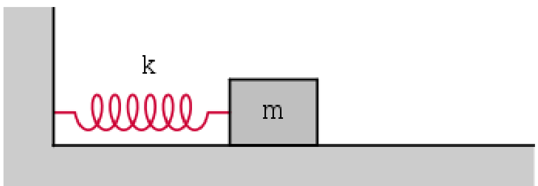
The pairs (e_i, f_i) of basis vectors are called **conjugate**. An isometry between symplectic vector spaces is called a (linear) **symplectomorphism**. The existence of symplectic bases shows that any two symplectic vector spaces of the same dimension are symplectomorphic. The group of isometries of a symplectic space (V, ω) with itself is called the **symplectic group** of (V, ω), and denoted $Sp(V, \omega)$.

Now let (v, w) be an arbitrary ordered pair of vectors in a symplectic space (V, ω). We spell out what $\omega(v, w)$ measures geometrically. Choose a symplectic basis and let $V = V_1 \oplus \cdots \oplus V_n$ be the corresponding direct sum decomposition given by setting $V_i = \text{span}(e_i, f_i)$ for each conjugate pair of basis vectors. Consider the parallelogram A spanned by v and w, and consider its projections onto each of the planes V_i. In other words, from (v, w) we obtain n parallograms A_i, spanned by the n pairs of vectors (v_i, w_i) which are the respective projections of v and w onto V_i. Then the number $\omega(v, w)$ is the sum of the signed areas of the parallelograms A_i. The signs here encode whether the vectors (v_i, w_i) are positively or negatively oriented with respect to the basis (e_i, f_i) of V_i. Since $\omega(v, w)$ is defined independently of any basis, this geometric description holds for *any* choice of symplectic basis.

5.2. Hamiltonian mechanics

The definition of a symplectic geometry may seem a natural thing to study from a pure mathematical point of view, but it also has plenty of physical relevance, although this is slightly more hidden than the immediate relevance of such notions of angle and length. Indeed, symplectic geometry first arose in the context of Hamiltonian mechanics, which is a major branch of classical mechanics, i.e. the study of systems of classical point particles (where a “point” can represent something very large, such as a planet). We explain this connection now briefly.[2]

An important, very simple example of a classical dynamical system is the situation where one has a mass m (modeled as a point particle) which is positioned on a horizontal surface and attached to a (horizontally positioned) coil spring, which is itself attached to a wall.



We denote by x the distance of the mass from the wall, and consider the following problem: assuming that we know the position $x(t_0)$ and velocity $\dot{x}(t_0)$ of the mass at some specific time t_0, we wish to describe the future motion of the mass via a function $x(t)$ of time. For simplicity, we only consider the interaction between the mass and the spring, i.e. we neglect friction, gravity, etc.. It turns out that a good model for this situation is given by the differential equation:

$$m\ddot{x}(t) = -kx(t) \tag{210}$$

[1]This is not unique – there are in general many such bases.

[2]We take an efficient route, rather than one which follows a historical or conceptual ordering.

where k is an empirically determined constant (a real number) which describes how "stiff" the spring is. Together with the initial data $x(t_0) = x_0$ and $\dot{x}(t_0) = v_0$, the equation (210) defines a so-called "initial value problem", a solution $x(t)$ of which describes the motion of the mass m as a function of time. The equation (210) is a description of the system in a "Newtonian formulation" – the mass times the acceleration of a particle is set equal to some description of a force acting on that particle (in this case the force exerted by the spring). We wish to transform this equation into a "Hamiltonian formulation" in order to illustrate the latter. For this, we first introduce new coordinates to describe the system at hand: we set

$$q(t) := x(t) \quad \text{and} \quad p(t) := m\dot{x}(t).$$

So q describes position and p describes momentum. This allows us to write the second-order differential equation (210) as a system of first-order differential equations

$$\dot{q} = \tfrac{1}{m}p$$
$$\dot{p} = -kq.$$

We think of $(q(t), p(t))$ as a vector in $\mathbb{R}^2$ (for each t) and we rewrite these two equations as

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial p}H(q,p) \\ -\frac{\partial}{\partial q}H(q,p) \end{bmatrix} \tag{211}$$

where H is the function $\mathbb{R}^2 \to \mathbb{R}$ defined by

$$H(q,p) = \tfrac{1}{2}kq^2 + \tfrac{1}{2}\tfrac{1}{m}p^2. \tag{212}$$

This is called the **Hamiltonian** function of the system, and (211) are **Hamilton's equations**. The right-hand side of (211) defines a vector field X_H on $\mathbb{R}^2$, i.e. a (smooth) map $\mathbb{R}^2 \to \mathbb{R}^2$; it is called the **Hamiltonian vector field** associated to H since

$$X_H = \begin{bmatrix} \frac{\partial}{\partial p}H(q,p) \\ -\frac{\partial}{\partial q}H(q,p) \end{bmatrix} \tag{213}$$

is completely determined by H.

Symplectic geometry enters now in how we view the process of getting X_H from H. Recall that to a bilinear form ω on V (in our example $V = \mathbb{R}^2$), we have the associated map

$$\tilde{\omega} : V \to V^*, \tilde{\omega}(v)(w) := \omega(v,w),$$

which is an isomorphism when ω is non-degenerate. The standard basis on $V = \mathbb{R}^2$ is a symplectic basis for the symplectic form ω given, in these coordinates, by the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

We may rewrite (222) as

$$X_H = \tilde{\omega}^{-1}dH, \tag{214}$$

where

$$dH : V \longrightarrow V^*, \begin{bmatrix} q \\ p \end{bmatrix} \longmapsto \begin{bmatrix} \frac{\partial}{\partial q}H(q,p) \\ \frac{\partial}{\partial p}H(q,p) \end{bmatrix}$$

is the differential of H. Indeed, the coordinate matrix of $\tilde{\omega}^{-1}$ with respect to the standard basis of V and its dual basis is (also) the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and so

$$[\tilde{\omega}^{-1} \circ dH] \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial q} H(q,p) \\ \frac{\partial}{\partial p} H(q,p) \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial p} H(q,p) \\ -\frac{\partial}{\partial q} H(q,p) \end{bmatrix}.$$

In principle, any (smooth) function $H : V \to \mathbb{R}$ may be taken to be a "Hamiltonian function", though it may not necessarily have physical relevance. A vector field $X : V \to V$ is called a **Hamiltonian vector field** if there exists some function H such that $X = X_H$ in the sense of (214). We note that not every vector field on V will be Hamiltonian.

One of the major benefits of the Hamiltonian formulation is that the description of a dynamical system is interlinked with a geometric theory. This allows for various simplifications and insights. Although we will not enter much further into this story, we indicate a few basic aspects.

The collection of all "possible" pairs (q, p) of position and momenta form what in classical mechanics is called **phase space**. In our simple example above, we took phase space to be $V = \mathbb{R}^2$ (although we might also have chosen a subset thereof). In Hamiltonian mechanics, phase space is modelled as a symplectic manifold or, more generally, as a Poisson manifold.[3] For purposes of illustration, we stick with our example $V = \mathbb{R}^2$, which is a simple special case. We say that the triple (V, ω, H) defines a **Hamiltonian system**. For each $(q_0, p_0) \in V$, we have an associated initial value problem

$$\begin{cases} \dot{c}(t) = X_H(c(t)) \\ c(t_0) = (q_0, p_0) \end{cases} \tag{215}$$

a solution of which is a (differentiable) curve $c : \mathbb{R} \to V$ (for simplicity, we neglect analytical subtleties, such as the fact that c might not be defined on all of $\mathbb{R}$). Under suitable conditions, this initial value problem will have a unique solution and there is a **flow map** associated to the vector field X_H: it is

$$\varphi : V \times \mathbb{R} \longrightarrow V, (q_0, p_0, t) \longmapsto c(t).$$

where $c(t) \in V$ is the value, at time t, of the unique solution c of (215). In other words φ describes the time evolution of a dynamical system, as dependent on initial positions and momenta. For each fixed t, the associated map

$$\varphi_t : V \longrightarrow V,\ (q_0, p_0) \longmapsto c(t)$$

is a symplectomorphism, i.e. the time evolution respects the geometry of phase space.

Another role that symplectomorphisms play in the Hamiltonian formulation is that of *symmetries*; we will focus here only on linear symmetries. Before making a definition of linear symmetry, we first give a characterization of linear symplectomorphisms in terms of their interaction with Hamiltonian systems. Namely, fix a real symplectic vector space (V, ω) and for every smooth function $H : V \to \mathbb{R}$, consider its associated Hamiltonian vector field X_H.

[3] A symplectic manifold is a (smooth) manifold equipped with a closed non-degenerate differential 2-form ω.

LEMMA 5.2.1. *Let $S : V \to V$ be an invertible linear map. Then S is a symplectomorphism if and only if*

$$X_{H\circ S} = S^{-1} X_H S \tag{216}$$

for any smooth function $H : V \to \mathbb{R}$.

PROOF. Note that

$$d(H \circ S) = S^* dHS \tag{217}$$

as maps $V \to V^*$. Indeed, given $v, w \in V$,

$$[d(H \circ S)(v)](w) = dH(Sv)(dS(v)w) = dH(Sv)(Sw) = [(S^* dHS)(v)](w)$$

by the chain rule and the fact that S is linear. Equivalently, this means that

$$\omega(X_{H\circ S} v, w) = \omega(X_H Sv, Sw) \qquad \forall v, w \in V. \tag{218}$$

Suppose now that S is symplectomorphism. Then we have

$$X_{H\circ S} = \tilde{\omega}^{-1} d(H \circ S) = \tilde{\omega}^{-1} S^* dHS = S^{-1} \tilde{\omega}^{-1} dHS = S^{-1} X_H S,$$

using that $\tilde{\omega}^{-1} S^* = S^{-1} \tilde{\omega}^{-1}$ since S is a symplectomorphism: indeed, it is readily checked that a linear map $S : V \to V$ is a symplectomorphism if and only if the diagram

$$\begin{array}{ccc} V & \xrightarrow{\tilde{\omega}} & V^* \\ {\scriptstyle S}\big\downarrow & & \big\uparrow{\scriptstyle S^*} \; x \\ xV & \xrightarrow[\tilde{\omega}]{} & V^* \end{array} \tag{219}$$

commutes.

Conversely, now suppose that S is an invertible linear map satisfying (216) for any H. Then, for any $v, w \in V$,

$$\omega(SX_{H\circ S} v, Sw) = \omega(X_S v, Sw) = \omega(X_{H\circ S} v, w) \tag{220}$$

holds, where the second equation is given by (218). To show that S is a symplectomorphism, we show that $X_{H\circ S} v$ can take any value in V as we vary H. Indeed, given $u \in V$ arbitrary, define $H := (\tilde{\omega}(u) \circ S^{-1})$, which is a linear function $V \to \mathbb{R}$. Then $dH : V \to V^*$ is the constant map with value $\tilde{\omega}(u) \circ S^{-1} \in V^*$, and so

$$X_{H\circ S} v = \tilde{\omega}^{-1} d(H \circ S) v = \omega^{-1} S^* dHSv = \omega^{-1} S^* \tilde{\omega}(u) \circ S^{-1} = u, \tag{221}$$

as desired. □

By a (linear) **symmetry** of a Hamiltonian system (V, ω, H) we mean an invertible linear map $S : V \to V$ such that if c is a solution of $\frac{d}{dt} c(t) = X_H(c(t))$, then $S \circ c$ is a solution of $\frac{d}{dt}(S \circ c)(t) = X_H((S \circ c)(t))$. We think of the invertible map S as corresponding to a change of coordinates, and a symmetry is a change of coordinates which does not change the form of Hamilton's equations (211).

PROPOSITION 5.2.2. *Let (V, ω, H) be a Hamiltonian system. If $S : V \to V$ is a linear symplectomorphism such that $H \circ S = H$, then S is a symmetry of (V, ω, H).*

PROOF. First, we note that $H \circ S = H$ implies that also $H \circ S^{-1} = H$. Now we compute

$$\begin{aligned} \frac{d(Sc)}{dt}(t) &= S\frac{dc}{dt}(t) = SX_H(c(t)) = SX_H(S^{-1}Sc(t)) \\ &= SX_H S^{-1}(Sc(t)) = X_{H\circ S^{-1}}(Sc(t)) = X_H(Sc(t)), \end{aligned}$$

which shows that S is a symmetry. □

REMARK 5.2.3. The diagramatic encoding (219) is useful for seeing at once various equivalent ways to express that a linear map S is a symplectomorphism; also, as explained in the introduction of this thesis, this kind of description is basic to our overall point view on geometric structures.

A vector field $X : V \to V$ is **linear** if it is a linear map. In Section 6.1 below we study linear hamiltonian vector fields. In general, a (smooth) vector field $X : V \to V$ need not, of course, be linear. However, in various examples, the "physical" Hamiltonian function is quadratic – e.g. as above in (212) – and so the associated vector field is indeed linear. In cases when X is non-linear, via Taylor expansion in points $v \in V$ where $X(v) = 0$, we may still approximate X locally with the help of a linear vector field, providing a first step in analyzing a dynamical system.

LEMMA 5.2.4. *Let (V, ω) be a symplectic vector space. A linear vector field $X : V \to V$ is Hamiltonian if and only if X is "symplectically skew self-adjoint" in the sense that the diagram*

$$\begin{array}{ccc} V & \xrightarrow{\tilde{\omega}} & V^* \\ {\scriptstyle X}\downarrow & & \downarrow{\scriptstyle -X^*} \\ V & \xrightarrow[\tilde{\omega}]{} & V^* \end{array} \tag{222}$$

commutes. In this case, $H(v) := \frac{1}{2}\omega(Xv, v)$ is a Hamiltonian function for X.

PROOF. Assume first that $X = X_H$ is Hamiltonian, for some function H. By definition, this means that

$$\tilde{\omega}X = dH. \tag{223}$$

The left-hand side of this equation is a linear function, so $dH : V \to V^*$ must be linear as well. This implies that $d^2H : V \to \mathrm{Hom}(V, V^*)$ is constant, with image $dH \in \mathrm{Hom}(V, V^*)$. On the other hand, because mixed partial derivatives commute, $d^2H(v) \in \mathrm{Hom}(V, V^*) \simeq \mathrm{Hom}(V \otimes V, \mathbb{R})$ is symmetric for each $v \in V$. Thus we find that $d^2H(v) = dH \in \mathrm{Hom}(V, V^*)$ corresponds to a symmetric bilinear form on V, i.e. $dH = (dH)^* \circ \iota$, where $\iota : V \to V^{**}$ is the usual canonical isomorphism. Using the relation (223), and the skew-symmetry of ω, this means that

$$\tilde{\omega}X = (\tilde{\omega}X)^* \circ \iota = X^*\tilde{\omega}^*\iota = -X^*\tilde{\omega}, \tag{224}$$

which is precisely the relation (222) to be shown.

Now assume that (222) holds, and set $H(v) := \frac{1}{2}\omega(Xv, v)$. Then we have, for $v, w \in V$,

$$\begin{aligned} (dHv)(w) &= \tfrac{1}{2}\omega(Xw, v) + \tfrac{1}{2}\omega(Xv, w) = \tfrac{1}{2}(\tilde{\omega}Xw)(v) + \tfrac{1}{2}(\tilde{\omega}Xv)(w) \\ &= -\tfrac{1}{2}(X^*\tilde{\omega}w)(v) + \tfrac{1}{2}(\tilde{\omega}Xv)(w) = -\tfrac{1}{2}(\tilde{\omega}w)(Xv) + \tfrac{1}{2}(\tilde{\omega}Xv)(w) \\ &= \tfrac{1}{2}(\tilde{\omega}Xv)(w) + \tfrac{1}{2}(\tilde{\omega}Xv)(w) = (\tilde{\omega}Xv)(w). \end{aligned}$$

□

CHAPTER 6

Linear Hamiltonian vector fields and symplectomorphisms

Throughout this chapter we work with finite-dimensional vector spaces over a field which is perfect[1] and not of characteristic 2.

We consider Section 6.1 below on linear Hamiltonian vector fields especially important from an expository point of view. In it one may see most of the essential techniques and ideas that will also appear later in the more involved analysis of isotropic triples, which is given in Chapter 7. It also serves as a template for a similar analysis which may be carried out for the case of linear symplectomorphisms. Instead of spelling out such a treatment, we have sketched a shortcut route and summarized the key results for linear symplectomorphisms in Section 6.2.

In Section 6.1, various of the Propositions and proofs are essentially duplicates of corresponding statements in Chapter 7. In such cases, we often omit the proof and refer to the corresponding statement there.

6.1. Linear Hamiltonian vector fields

As discussed in Lemma 5.2.4, a linear Hamiltonian vector field on a symplectic vector space (V,ω) is a linear map $X : V \longrightarrow V$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\tilde{\omega}} & V^* \\ {\scriptstyle X}\downarrow & & \downarrow{\scriptstyle -X^*} \\ V & \xrightarrow[\tilde{\omega}]{} & V^* \end{array} \tag{225}$$

commutes. We can change perspective slightly, and view this data as consisting of the pair (V, X) – an endomorphism of a vector space V – together with a symplectic structure ω which is compatible with X in the sense that (225) commutes. From this perspective, we are precisely in the situation of Example 3.2.5, with $\varepsilon = -1$. That is, a linear Hamilionian vector field (V, X, ω) is the same thing as a fixed point in the category $\mathrm{End}(\mathsf{vect}^-)$ whose objects are endomorphisms (V, X) and which is equipped with the duality involution (δ, η) where

$$\delta(V, X) = (V^*, -X^*)$$

and

$$\eta_V : V \longrightarrow V^{**}, v \longmapsto (\xi \mapsto -\xi(v)).$$

Thus, given a linear Hamiltonian vector field (V, X, ω), we call (V, X), or just X, the **underyling endomorphism**. In fact, we will use here various terms from Chapter 4 (and also Sections 2.3.3 and 3.2.8) which apply to the special case of linear Hamiltonian vector fields: terms such as "**morphism**", "**indecomposable**", "**compatible form**", etc.. So, for example, a morphism

$$(V, X, \omega) \longrightarrow (V', X', \omega')$$

[1]A field $\mathbf{k}$ is called **perfect** if every algebraic extension of $\mathbf{k}$ is separable. There are many fields which fulfil this condition: examples of perfect fields include all finite fields, and all fields of characteristic zero.

of linear Hamiltonian vector fields is a linear map which is a morphism $(V, X) \to (V, X')$ of endomorphisms and which is also an isometry $(V, \omega) \to (V', \omega')$.

In the subsequent subsections, we will employ the results of Chapter 4 toward classifying, up to isomorphism, all indecomposable linear Hamiltonian vector fields. For the case $\mathbf{k} = \mathbb{R}$ as an illustrative example, we will obtain a full classification. The basic "recipe" is as follows.

REMARK 6.1.1 (Classification procedure).

Step 1 Start with a description of the indecomposable types for the underlying category. Here the underlying category is $\mathsf{C} = \mathrm{End}(\mathsf{vect}_{\mathbf{k}})$.

Step 2 Classify which indecomposables are dual to which; in particular, find the self-dual ones.

Step 3 Determine which self-duals admit compatible forms.

Step 4 For each self-dual indecomposable admitting a compatible form, determine how many such forms exist, up to isometry.

6.1.1. Indecomposable endomorphisms. As regards Step 1, we first recall the definition of the category $\mathsf{End}(\mathsf{vect}_{\mathbf{k}})$. Its objects are pairs (V, X) consisting of a finite-dimensional $\mathbf{k}$-vector space and an endomorphism $X : V \to V$. A morphism $S : (V, X) \to (V', X')$ is a linear map $S : V \to V'$ such that

$$\begin{array}{ccc} V & \xrightarrow{X} & V \\ \big\downarrow S & & \big\downarrow S \\ V' & \xrightarrow{X'} & V' \end{array} \tag{226}$$

commutes. $\mathsf{End}(\mathsf{vect}_{\mathbf{k}})$ is an additive category, where the biproduct is the obvious notion of direct sum of endomorphisms:

$$(V, X) \oplus (V, X') := (V \oplus V', X \oplus X'). \tag{227}$$

Furthermore, the Krull-Schmidt theorem holds in $\mathsf{End}(\mathsf{vect}_{\mathbf{k}})$, i.e. every endomorphism (V, X) has a direct sum decomposition into indecomposable summands, and such a decomposition is essentially unique. It is a general fact of linear algebra that, up to isomorphism, the indecomposable endomorphisms are enumerated by objects of the following form

$$(\mathbf{k}[t]/(p^m), M_{\underline{t}}) \tag{228}$$

where p ranges over all monic irreducible polynomials in $\mathbf{k}[t]$, m ranges over all positive integers, and $M_{\underline{t}}$ is the endomorphism "multiplication by $\underline{t}$", where $\underline{t}$ is the image of $t \in \mathbf{k}[t]$ under the quotient map $\mathbf{k}[t] \to \mathbf{k}[t]/(p^m)$. The vector space $\mathbf{k}[t]/(p^m)$ comes with a canonical ordered basis

$$(\underline{1}, \underline{t}, \underline{t}^2, ..., \underline{t}^{km-1}), \tag{229}$$

where $k = \deg(p)$. This exhibits in particular that $(\mathbf{k}[t]/(p^m), M_{\underline{t}})$ is cyclic, i.e. there exists a non-zero vector whose orbit under the action of $M_{\underline{t}}$ is the whole vector space (here, $\underline{1}$ is such a vector).

For a given indecomposable $(\mathbf{k}[t]/(p^m), M_{\underline{t}})$, the polynomial p^m is the minimal polynomial of $M_{\underline{t}}$, and this polynomial is also the characteristic polynomial, since $M_{\underline{t}}$ is indecomposable. Thus the list (228) says that the isomorphism class of an *indecomposable* endomorphism is completely characterized by its minimal (= characteristic) polynomial.

In the following it will be useful to have normal forms, in terms of coordinate matrices, for indecomposable endomorphisms. We will work with a generalized Jordan normal form (which we call Mal'cev normal form) which always exists provided that we work over a ground field which is perfect.

Over an algebraically closed field, the usual Jordan normal form of an indecomposable endomorphism (V, X) amounts to the existence of a basis of V such that the associated coordinate matrix of X has a single eigenvalue as its diagonal entries, ones as entries on the upper diagonal, and zeros as all other entries. The generalized Jordan normal form will be a block matrix having a similar form as before, but with the eigenvalue replaced by a suitable generalization.

To formulate this, recall that given a monic polynomial

$$q(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0,$$

its **companion matrix** (also called "Frobenius matrix") is the matrix

$$\begin{bmatrix} 0 & \dots & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & 1 & 0 & -a_{n-2} \\ 0 & \dots & 0 & 1 & -a_{n-1} \end{bmatrix}. \tag{230}$$

This is precisely the coordinate matrix of the endomorphism $(\mathbf{k}[t]/(p^m), M_{\underline{t}})$ with respect to the basis (229), with $q(t) = p(t)^m$.

PROPOSITION 6.1.2 (Mal'cev normal form). *Let* $\mathbf{k}$ *be a perfect field, and let* (V, X) *be an indecomposable in* $\mathsf{End}(\mathsf{vect}_{\mathbf{k}})$. *Let* $q(t) = p(t)^m$ *be the minimal polynomial of* X, *with* $p(t)$ *irreducible. There exists a basis of* V *with respect to which the coordinate matrix of* X *has the form*

$$\begin{bmatrix} Z & \mathbb{1} & 0 & \dots & 0 \\ 0 & Z & \mathbb{1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbb{1} & 0 \\ & & 0 & Z & \mathbb{1} \\ 0 & \dots & \dots & 0 & Z \end{bmatrix} \tag{231}$$

where Z *is the companion matrix of* $p(t)$.

For proofs and discussion, see [**Mal63**]. In the case of an indecomposable endomorphism of the form $(\mathbf{k}[t]/(p^m), M_{\underline{t}})$, a basis putting $M_{\underline{t}}$ into Malcev normal form is

$$\{\underline{1}, \underline{t}, \underline{t}^2, ..., \underline{t}^{m-1}; \underline{1}p(\underline{t}), \underline{t}p(\underline{t}), \underline{t}^2 p(\underline{t}), ..., \underline{t}^{m-1}p(\underline{t}); \dots$$
$$\dots \underline{1}p(\underline{t})^{k-1}, \underline{t}p(\underline{t})^{k-1}, \underline{t}^2 p(\underline{t})^{k-1}, ..., \underline{t}^{k-1}p(\underline{t})^{m-1}\},$$

(see, for example, [**Rob70**]).

6.1.2. Self-dual indecomposables. We proceed to Step 2: we wish to determine which indecomposable endomorphisms are dual to which, with respect to our chosen duality involution. Namely, the dual of an endomorphism (V, X) is $(V^*, -X^*)$. Since this duality involution is an additive functor, the dual $(V^*, -X^*)$ is indecomposable when (V, X) is. Thus, to identify which indecomposable endomorphisms are dual to each other, we need to know the minimal polynomial of $-X^*$ relates to the minimal polynomial of X.

Consider the following involution on $\mathbf{k}[t]$: given a polynomial

$$q(t) = a_k t^k + a_{k-1}t^{k-1} + \cdots + a_1 t + a_0,$$

set

$$q^\dagger(t) := a_k t^k + (-1)^1 a_{k-1} t^{k-1} + \cdots + (-1)^{k-1} a_1 t + (-1)^k a_0.$$

Note that if $q(t)$ is monic, then so is $q^\dagger(t)$.

LEMMA 6.1.3. *Let $q, p \in \mathbf{k}[t]$. Then $(qp)^\dagger = q^\dagger p^\dagger$.*

PROOF. Let $p = (a_k, a_{k-1}, ..., a_1, a_0)$ and $q = (b_l, b_{l-1}, ..., b_1, b_0)$ and recall that the n-th coefficient of pq is $c_n = \sum_{i+j=n} a_i b_j$. The n-th coefficient of $(pq)^\dagger$ can be written as $(-1)^{n-(k+l)} c_n = \sum_{i+j=n} (-1)^{n-k-l} a_i b_j$.

On the other hand, the n-th coefficient of $p^\dagger q^\dagger$ is

$$\sum_{i+j=n} (-1)^{i-k} a_i (-1)^{j-k} b_j = \sum_{i+j=n} (-1)^{i+j-k-l} a_i b_j = \sum_{i+j=n} (-1)^{n-k-l} a_i b_j.$$

□

LEMMA 6.1.4. *Let (V, X) be an endomorphism, and let q_X be the minimal polynomial of X, and q_{-X} the minimal polynomial of $(V, -X)$. Then*

$$q_{-X} = q_X^\dagger.$$

PROOF. Note, first of all, that for any polynomial $p = (a_k, ..., a_0)$ it holds that $p^\dagger(-T) = (-1)^{\deg(p)} p(T)$:

$$\begin{aligned} p^\dagger(-T) &= a_k(-T)^k + (-1)^1 a_{k-1}(-T)^{k-1} ... + (-1)^{k-1} a_1 (-T)^1 + (-1)^k a_0 \\ &= (-1)^k p(T). \end{aligned}$$

It follows in particular that $-T$ is annihilated by $q_X^\dagger$:

$$q_X^\dagger(-T) = (-1)^{\deg(q_X)} q_X(T) = 0.$$

This shows that $q_{-X} | q_X^\dagger$, and so $\deg(q_{-X}) \le \deg(q_X^\dagger) = \deg(q_X)$.

On the other hand,

$$q_{-X}^\dagger(T) = q_{-X}^\dagger(-(-T)) = (-1)^k q_{-X}(-T) = 0,$$

which implies that $q_X | q_{-X}^\dagger$. In particular $\deg(q_X) \le \deg(q_{-X}^\dagger) = \deg(q_{-X})$. Thus we find that $\deg(q_{-X}) = \deg(q_X^\dagger)$, and so $q_{-X} = q_X^\dagger$. □

LEMMA 6.1.5. *Let (V, X) be an endomorphism. Then (V, X) is isomorphic (non-canonically) to (V^*, X^*) via a linear isomorphism $T : V \to V^*$ which is symmetric in the sense that $(Tv)(w) = (Tw)(v)$ for all $v, w \in V$.*

PROOF. See, for instance, [**TZ59**]. □

COROLLARY 6.1.6. *Let (V, X) be an endomorphism, with minimal polynomial q_X. Then $(V^*, -X^*)$ has minimal polynomial*

$$q_{-X^*} = q_X^\dagger.$$

In particular, if (V, X) is self-dual, i.e. if (V, X) is isomorphic to $(V^*, -X^*)$, then

$$q_X = q_X^\dagger.$$

PROOF. By Lemma 6.1.5, $(V, -X)$ and $(V, -X^*)$ are isomorphic, so they have the same minimal polynomial. And by Lemma 6.1.4 this polynomial is $q_X^\dagger$, where q_X is the minimal polynomial of (V, X). □

COROLLARY 6.1.7. *Let* $(V, X) = (\mathbf{k}[t]/(p^m), M_{\underline{t}})$ *be an indecomposable endomorphism (so p is irreducible). Then* (V, X) *is self-dual if and only if* $p = p^\dagger$*.*

PROOF. The minimal polynomial of $(V, X) = (\mathbf{k}[t]/(p^m), M_{\underline{t}})$ is $q_X = p^m$.

If (V, X) is self-dual, then by Corollary 6.1.6, $q_X = q_X^\dagger$. Conversely, if $q_X = q_X^\dagger$, then (V, X) is self-dual, since for (V, X) indecomposable, $(V^*, -X^*)$ is also indecomposable, and for indecomposable endomorphisms, the minimal polynomial is a complete invariant.

We need to still show that $q_X = q_X^\dagger$ if and only if $p = p^\dagger$. By Lemma 6.1.3, $q_X^\dagger = (p^m)^\dagger = (p^\dagger)^m$, so clearly $p = p^\dagger$ implies $q_X = q_X^\dagger$. On the other hand, if $q_X^\dagger = q_X$, then $q_X = (p^\dagger)^m = p^m$ are two factorizations of q_X as a product of irreducibles. Since such a factoriziation must be unique, $p = p^\dagger$ follows. □

The previous corollary tells us that the isomorphism classes of self-dual indecomposable endomorphisms are enumerated by pairs (p, m), where m is a positive integer and p is an irreducible monic polynomial such that $p = p^\dagger$.

LEMMA 6.1.8. *Suppose p is an irreducible monic polynomial such that* $p = p^\dagger$*. Then,*

Type I: $p(t) = t$, *or*

Type II: *Only even powers of t appear in* $p(t)$*. In particular,* $p(t)$ *has even degree.*

PROOF. We assume $p^\dagger = p$. Let $\deg p = 1$. Then $p(t) = t + a_0$ and $p^\dagger(t) = t - a_0$, so $p^t = p$ implies $a_0 = 0$.

Now suppose $\deg(p) > 1$. If $\deg(p)$ were odd then $p^\dagger = p$ implies that $(-1)^k a_0 = a_0$ means $-a_0 = a_0$, so $a_0 = 0$. But then t divides p, a contradiction to the irreducibility of p. This shows that $\deg(p)$ is even.

Now consider a_n, the n-th coefficient of of p, for n odd. The n-th coefficient of $p^\dagger$ is $(-1)^{k-n} a_n$, where $k = \deg(p)$. Since $p^\dagger = p$, we have $(-1)^{k-n} a_n = a_n$, and since k is even, $k - n$ is odd, so we must have $a_n = 0$. □

REMARK 6.1.9. Let $p(t) = t$. The indecomposable endomorphisms of the type

$$(\mathbf{k}[t]/(p^m), M_{\underline{t}})$$

are precisely the indecomposable nilpotent ones.

In general, the question of which polynomials in $\mathbf{k}[t]$ are irreducible is, of course, dependent on the field $\mathbf{k}$. For algebraically closed fields, irreducible polynomials have degree 1, so the only self-dual irreducible monic polynomial in this case is $p(t) = t$. For $\mathbf{k} = \mathbb{R}$, the irreducible monic polynomials are

$$p(t) = t + a_0, \quad a_0 \in \mathbb{R} \qquad \text{and} \qquad p(t) = t^2 - 2xt + (x^2 + y^2), \quad x \in \mathbb{R}, y > 0. \tag{232}$$

In other words, these irreducibles are parametrized by the orbits of the "complex conjugation action" of $\mathbb{Z}_2$ on $\mathbb{R}^2 \simeq \mathbb{C}$ given by $(x, y) \mapsto (x, -y)$. We are folding the plane along the real axis. The involution $p \mapsto p^\dagger$ induces an action of $\mathbb{Z}_2$ on the above space of irreducible polynomials which corresponds to $(x, \pm y) \mapsto (-x, \pm y)$, i.e. we are folding along the imaginary axis. Fixed points of this action correspond to the irreducibles that are self-dual; these are

$$p(t) = t \qquad \text{and} \qquad p(t) = t^2 + a_0, \quad a_0 > 0. \tag{233}$$

For reference later, we summarize our result for $\mathbf{k} = \mathbb{R}$:

COROLLARY 6.1.10. *The self-dual indecomposable endomorphisms over* $\mathbf{k} = \mathbb{R}$ *are, up to isomorphism, those* $(\mathbb{R}[t]/(p^m), M_{\underline{t}})$ *for which p is one of the types (233).*

6.1.3. Existence of compatible forms. We are now at Step 3 of the classification procedure given in Remark 6.1.1.

Let $(V, X) = (\mathbf{k}[t]/(p^m), M_{\underline{t}})$ be a self-dual indecomposable endomorphism. From the previous section we know that its minimal polynomial p^m is such that $p^\dagger = p$, which means that p is an irreducible polynomial which is either $p(t) = t$ (type I) or such that p only contains even powers of t (type II). Note that in the latter case, p^m will also only contain even powers of t. Our aim in this section is to tackle "Step 3", i.e. to determine skew-symmetric compatible forms exist for (V, X).

As a first step, we describe an explicit symmetric isomorphism $T : (V, X) \to (V^*, X^*)$ which works for any indecomposable (V, X). Then we'll define an isomorphism $(V, X) \to (V, -X)$, using that (V, X) is self-dual.

Let $k = \deg(p)$, and set $n = km$. Let

$$q(t) := p^m(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0. \tag{234}$$

Consider the basis

$$(\underline{1}, \underline{t}, \underline{t}^2, ..., \underline{t}^{n-1}) \tag{235}$$

of V and the linear map $\tau : V \to \mathbf{k}$ defined by

$$\tau(\underline{t}^j) = \begin{cases} 1 & \text{if } j = n-1 \\ 0 & \text{if } j < n-1. \end{cases} \tag{236}$$

Recall that by definition

$$q(\underline{t}) = \underline{t}^n + a_{n-1}\underline{t}^{n-1} + \cdots + a_1\underline{t} + a_0 = 0,$$

so

$$\underline{t}^n = -a_0 - a_1\underline{t} - \cdots - a_{n-1}\underline{t}^{n-1}.$$

In particular,

$$\tau(\underline{t}^n) = \tau(-a_0 - a_1\underline{t} - \cdots - a_{n-1}\underline{t}^{n-1}) = -a_{n-1},$$

and

$$\tau(\underline{t}^{n+1}) = \tau(\underline{t}^n\underline{t}) = \tau(-a_0\underline{t} - a_1\underline{t}^2 - \cdots - a_{n-2}\underline{t}^{n-1} - a_{n-1}\underline{t}^n) = -a_{n-2} - a_{n-1}\tau(\underline{t}^n),$$

and so on.

Now we define $T : V \to V^*$ via the basis (235) by

$$(T\underline{t}^j)(\underline{t}^l) := \tau(\underline{t}^{j+l}) \qquad 0 \leq j, l \leq n-1. \tag{237}$$

Lemma 6.1.11. *$T : V \to V^*$ is a symmetric isomorphism.*

Proof. It is immediate from (237) - since addition of integers is commutative - that T is symmetric.

To see that T is invertible, note that, by (236), $(T\underline{t}^j)(\underline{t}^l) = 0$ if $i + j < n - 1$ and $(T\underline{t}^j)(\underline{t}^l) = 1$ if $i + j = n - 1$. Thus the coordinate matrix of T (with respect to the above basis) is of the form

$$\begin{bmatrix} & & & & 1 \\ & & & 1 & * \\ & & \iddots & \iddots & \vdots \\ & 1 & \iddots & & \\ 1 & * & \dots & & * \end{bmatrix}, \tag{238}$$

with zeros above the anti-diagonal. Using the Laplace expansion formula for determinants, it follows that $\det T = \pm 1$, so T is an isomorphism. $\square$

LEMMA 6.1.12. *T is a morphism $(V,X) \to (V^*, X^*)$.*

PROOF. We check this using the basis (235). On the one hand,

$$(TX\underline{t}^j)(\underline{t}^l) = (T\underline{t}^{j+1})(\underline{t}^l) = \tau(\underline{t}^{j+l+1}).$$

On the other hand,

$$(X^*T\underline{t}^j)(\underline{t}^l) = (T\underline{t}^j)(X\underline{t}^l) = (T\underline{t}^j)(\underline{t}^{l+1}) = \tau(\underline{t}^{j+l+1}).$$

$\square$

Now we construct an isomorphism $D : (V,X) \to (V,-X)$. Note that $-X = -M_{\underline{t}} = M_{-\underline{t}}$. Define D on the basis (235) by

$$D(\underline{t}^j) = (-1)\underline{t}^j \qquad\qquad 0 \leq j \leq km-1.$$

So, in coordinates, D is given by the matrix

(239)
$$\begin{bmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & (-1)^{km-1} \end{bmatrix}.$$

LEMMA 6.1.13. *Let $(V,X) = (\mathbf{k}[t]/(p^m), M_{\underline{t}})$ be indecomposable and self-dual. Then D defines an isomorphism*

$$(V,X) \to (V,-X).$$

PROOF. Clearly D is a linear isomorphism. To check that D intertwines X and $-X$, we use the basis (235). First let $j \in \{0,...,n-2\}$. We have

$$DX\underline{t}^j = D\underline{t}^{j+1} = (-1)^{j+1}\underline{t}^{j+1} = -(-1)^j\underline{t}^{j+1} = -(-1)^jX\underline{t}^j = -XD\underline{t}^j.$$

Now let $j = n-1$, and let $q = p^m = t^n + a_{n-1}t^{n-1} + \cdots + a_0$.
<u>Case 1</u>: $p(t) = t$. In this case $q(t) = t^n$, and so $\underline{t}^n = 0$. Thus we have

$$DX\underline{t}^j = D\underline{t}^{j+1} = 0$$

and

$$-XD\underline{t}^j = -X(-1)^j\underline{t}^j = (-1)^{j+1}\underline{t}^{j+1} = 0.$$

<u>Case 2</u>: p (and hence also $q = p^m$) contains only even powers of t. In particular $j = n-1$ is odd, since $n = \deg(q)$ is even.
Here, on the one hand, with $j = n-1$,

$$\begin{aligned} DX\underline{t}^j = D\underline{t}^n &= -a_0 - a_1D\underline{t} - a_2D\underline{t}^2 - \cdots - a_{n-1}D\underline{t}^{n-1} \\ &= -a_0 - (-1)^1\underline{t} - a_2(-1)^2\underline{t}^2 - \cdots - (-1)^{n-1}a_{n-1}\underline{t}^{n-1} \\ &= -a_0 - a_2\underline{t}^2 - \cdots - a_{n-2}\underline{t}^{n-2} \\ &= \underline{t}^n = \underline{t}^{j+1}. \end{aligned}$$

And on the other hand

$$-XD\underline{t}^j = -XD\underline{t}^{n-1} = -X(-1)^{n-1}\underline{t}^{n-1} = X\underline{t}^{n-1} = \underline{t}^n = \underline{t}^{j+1}.$$

since $n-1$ is odd. $\square$

PROPOSITION 6.1.14. *Let* $(V, X) = (\mathbf{k}[t]/(p^m), M_{\underline{t}})$ *be an indecomposable self-dual endomorphism, and let* $n = deg(p^m)$. *Define a bilinear form* B *on* V *via the basis (235) by*

$$B(\underline{t}^j, \underline{t}^l) = (-1)^j \tau(\underline{t}^{j+l}) \qquad\qquad 0 \leq j, l \leq n-1,$$

where $\tau : V \to \mathbf{k}$ *is the map (236), i.e.*

$$\tau(\underline{t}^j) = \begin{cases} 1 & \text{if } j = n-1 \\ 0 & \text{if } j < n-1. \end{cases}$$

Then B *defines a compatible form which is symmetric if* $dim V$ *is odd, and skew-symmetric if* $dim V$ *is even.*

PROOF. It is readily checked that B corresponds to the map $TD : V \to V^*$, where T is the map from Lemma 6.1.11 and 6.1.12, and D is the map from Lemma 6.1.13. This shows that B is a compatible form, since

$$(V, X) \xrightarrow{D} (V, -X) \xrightarrow{T} (V, -X^*).$$

Next, to show that B is pseudosymmetric, with parity opposite to the parity of $\dim V$, note that

$$B(\underline{t}^j, \underline{t}^l) = (-1)^j(-1)^{2l}\tau(\underline{t}^{j+l}) = (-1)^{j+l}(-1)^l\tau(\underline{t}^{j+l}) = (-1)^{j+l}B(\underline{t}^l, \underline{t}^j). \tag{240}$$

Now we consider two cases.

Case 1: $p(t) = t$. In this case, for any $h \in \mathbb{N}$, we have $\tau(\underline{t}^h) = 1$ if $h = n-1$, and $\tau(\underline{t}^h) = 0$ otherwise. Thus, when $j + l = n - 1$,

$$B(\underline{t}^j, \underline{t}^l) = (-1)^{j+l}B(\underline{t}^l, \underline{t}^j) = (-1)^{n-1}B(\underline{t}^l, \underline{t}^j).$$

And when $j + l \neq n - 1$, then $B(\underline{t}^j, \underline{t}^l)$ and $B(\underline{t}^l, \underline{t}^j)$ are in any case both zero.

Case 2: p (and so also $q = p^m$) contains only even powers of t. In particular $\dim V = \deg(q)$ is even. In this case, for any $h \in \mathbb{N}$, we have $\tau(\underline{t}^h) = 0$ if h is even. This is clear for $h \leq n$, and for $h > n$ may be proved via induction, since when n is even

$$\underline{t}^h = \underline{t}^n \underline{t}^{h-n} = (-a_0 - a_2\underline{t}^2 - \cdots - a_{n-1}\underline{t}^{n-2})\underline{t}^{h-n}$$

is again a sum of even powers of $\underline{t}$. Thus, for $j + l$ even, both $B(\underline{t}^j, \underline{t}^l)$ and $B(\underline{t}^l, \underline{t}^j)$ are zero. If $j + l$ is odd, then

$$B(\underline{t}^j, \underline{t}^l) = (-1)^{j+l}B(\underline{t}^l, \underline{t}^j) = -B(\underline{t}^l, \underline{t}^j),$$

as claimed.

□

REMARK 6.1.15. With respect to the basis (235), the coordinate matrix of the compatible bilinear form B has the following structure

$$\begin{bmatrix} & & & & (-1)^{n-1}1 \\ & & & \iddots & * \\ & & 1 & \iddots & \vdots \\ & -1 & \iddots & & \\ 1 & * & \ldots & & * \end{bmatrix}, \tag{241}$$

(here, "$*$" indicates unspecified elements of the matrix). In the special case when $q(t) = t^n$, then all entries below the antidiagonal are also zero.

6.1.4. Uniqueness of compatible forms. We come to Step 4 of the procedure listed in Remark 6.1.1. For the question of uniqueness of compatible forms, we assume now that we are working over a ground field $\mathbf{k}$ which is perfect. This allows to use the Mal'cev normal form discussed in Proposition 6.1.2.

In fact, we will first describe a way to construct compatible forms which uses this normal form, and is slightly different from the construction of the previous section. For this, let (V, X) be an indecomposable self-dual endomorphism, with minimal polynomial p^m, where p is irreducible and of degree l. Let Z be the companion matrix of p (it is an $l \times l$ matrix), and let A be the coordinate matrix of X in Mal'cev normal form, with respect to a suitable fixed basis of V.

LEMMA 6.1.16. *If l is even, the $l \times l$ companion matrix Z admits both symmetric and skew-symmetric compatible forms. That means: there exist both symmetric and skew-symmetric non-degenerate matrices T such that*

$$TZT^{-1} = -Z^t.$$

If l is odd, there exists compatible symmetric forms.

With this lemma we can prove

PROPOSITION 6.1.17. *Let A be coordinate matrix of the indecomposable endomorphism X in Mal'cev normal form, with block diagonal entries Z. Let T be a matrix as in Lemma 6.1.16, and define the following block matrix*

$$H = H_T = \begin{bmatrix} & & & & (-1)^{m-1}T \\ & & & \cdots & \\ & & T & & \\ & -T & & & \\ T & & & & \end{bmatrix} \tag{242}$$

(all entries away from the block-antidiagonal are zero). Then H defines a compatible form for A, i.e.

$$HAH^{-1} = -A^t$$

and the parity $\varepsilon(H)$ of H is determined by the parity of T via $\varepsilon(H) = (-1)^{m-1}\varepsilon(T)$.

COROLLARY 6.1.18. *Let (V, X) be an indecomposable self-dual endomorphism, with minimal polynomial p^m, where p is irreducible and of degree l. We continue to assume that the ground field is perfect.*

If p is of type I, i.e $p(t) = t$, then X admits compatible symmetric forms if m is odd, and compatible skew-symmetric forms if m is even.

If p is of type II, then $\dim(V) = ml$ is even and X admits both compatible symmetric and skew-symmetric forms.

REMARK 6.1.19. Note that if p is of type I, i.e $p(t) = t$, then T is a 1×1 matrix, and H coincides with the matrix given in Remark 6.1.15. If p is of type II, this will in general not be the case; in particular, for p of type II, the matrix in Remark 6.1.15 is always skew-symmetric.

PROOF OF THE COROLLARY. The statement for p of type I follows from the results of the previous section, as does the existence of skew-symmetric forms for p of type II.

For the existence of compatible symmetric forms in the case of p of type II, the integer l is necessarily even, and we consider two subcases. If m is odd, we choose T to be a symmetric compatible form as in Lemma 6.1.16, and if m is even, we choose T to be skew-symmetric. Either way then, via $\varepsilon(H) = (-1)^{m-1}\varepsilon(T)$, the parity of H will be even, i.e. H defines a compatible symmetric form. □

Proof of the Proposition. The proof is a calculation with block matrices. We refer to the proof of Proposition 8.6.9, where essentially the same proof is spelled out. □

Proof of the Lemma. The statement for l odd follows from the previous section, where existence of symmetric compatible forms in odd dimension was shown. For l even, this lemma is proved in Proposition 8.6.4. The proof given there is quite "hands-on" and technical; it should be possible to give a more conceptual proof using the fact that a self-dual monic polynomial p of type II, when we pass to an algebraic closure of the ground field, will be of the form $p = s^\dagger s$, for some polynomial s. The expectation is that for polynomials of this form, there should be a canonical way to construct compatible symmetric and skew-symmetric forms for the associated endomorphism. We leave this idea, however, for future work. □

As a next step toward studying uniqueness of compatible forms for an indecomposable self-dual endomorphism (V, X), we need a description of the endomorphism algebra of (V, X). By definition, this consists of all endomorphisms of V which commute with X.

We fix again a basis of V such that X is in Mal'cev normal form, with coordinate matrix A. We give a description of the endomorphism algebra of X in terms of the endomorphism algebra of A. Denote by N the standard nilpotent matrix having all entries zero, except for ones on the upper off-diagonal. As above, we let p^m be the minimal polynomial of X, with p irreducible of degree l. Thus the Mal'cev normal form is in terms of block-matrices with $l \times l$ blocks. Note that N^l, the l^{th} power of N, may be viewed as a "standard nilpotent block matrix", with $l \times l$ blocks.

The following proposition says that elements of $\mathrm{End}(A)$ are "polynomials in N^l with coefficients in $\mathbf{k}(Z)$". The field $\mathbf{k}(Z)$ is the subring of $ml \times ml$ matrices given by matrix polynomials in the block diagonal matrix having Z as its diagonal blocks. That this subring is actually a field follows from the fact that the minimal polynomial p of Z is irreducible. For notational convenience, we set $F := \mathbf{k}(Z)$.

Proposition 6.1.20. *Let A be in Mal'cev normal form, with diagonal blocks given by the companion matrix Z of the irreducible polynomial p of degree l. The endomorphism algebra $End(A)$ of A consists of the matrices of the form*

$$\sum_{i=0}^{k-1} Z_i (N^l)^i, \qquad Z_i \in F. \tag{243}$$

Furthermore, in (243) the "coefficients" Z_i are unique.

Given an arbitrary element $C = \sum_{i=0}^{k-1} Z_i (N^l)^i$ of $End(A)$, note that it is block upper triangular, with Z_0 as its block-diagonal entries. In particular C is invertible if and only if Z_0 is invertible.

The radical of $End(A)$ consists of those $\sum_{i=0}^{k-1} Z_i (N^l)^i$ for which $Z_0 = 0$; we have

$$End(A) = F\mathbb{1} \oplus Rad(End(A)).$$

Proof. This is treated in Proposition 8.6.11 and its proof. □

Now let us consider two compatible forms, B_1 and B_2, for the self-dual indecomposable endomorphism (V, X). We assume that they have the same parity, and we say that B_1

and B_2 are **equivalent** if there exists an endomorphism f of (V, X) which is an isometry between B_1 and B_2. In other works, such that $B_2(fv, fw) = B_1(v, w)$ for all $v, w \in V$. The question of uniqueness of compatible forms for (V, X) is the question of how many such forms (of a fixed parity) exist up to equivalence. We continue to work with a basis for which X has Mal'cev normal form A. Let H_1 and H_2 be the corresponding coordinate matrices of B_1 and B_2, respectively.

LEMMA 6.1.21. *Let A be the Mal'cev matrix of a self-dual indecomposable endomorphism, with diagonal blocks given by the companion matrix Z. Let H_1 and H_2 be two compatible forms for A of the same parity (i.e. either both symmetric or both skew-symmetric). Then there exists an invertible "scalar" $C_0 \in F = \mathbf{k}(Z)$ such that H_2 and $H_1C_0\mathbb{1}$ are equivalent.*

REMARK 6.1.22. Note that, for $C_0 \in F$ invertible, $C_0\mathbb{1}$ is in the center of $\mathrm{End}(A)$, so H_1Z is again a compatible form.

PROOF. The above lemma and the proof given here are a special case of Lemma 8.6.16 and its proof. Nevertheless, we include this proof also here to show the reader what it involves. In particular it relies a bit on some algebraic trickery which, to me, is still somewhat opaque. We set $E := \mathrm{End}(A)$ and we let ε denote the parity of the compatible forms, i.e. $H_1^t = \varepsilon H_1$.

Let † denote the antiautomorphism given by the operation of adjoint with respect to H_1, i.e. $M^\dagger = H_1^{-1}M^tH_1$ for any matrix M (of the correct size). Note that when M is in the endomorphism algebra E of A, then so is $M^\dagger$. Note also that $(H_1^{-1})^\dagger = H_1^{-t}$.

Observe that $H_1^{-1}H_2$ determines an automorphism of A, so

$$H_1^{-1}H_2 = C_0I - R_0 \text{ for some invertible } C_0 \in F \text{ and } R_0 \in \mathsf{rad}E.$$

It follows that

$$\begin{aligned}(C_0I)^\dagger - R_0^\dagger &= (H_1^{-1}H_2)^\dagger = H_2^\dagger(H_1^{-1})^\dagger = H_1^{-1}H_2^tH_1H_1^{-t}\\ &= H_1^{-1}\varepsilon H_2H_1\varepsilon H_1^{-1} = H_1^{-1}H_2 = C_0I - R_0.\end{aligned}$$

Since $E = FI \oplus \mathrm{rad}\, E$ and this decomposition is preserved under taking adjoints, $(C_0I)^\dagger = C_0I$ and $R_0^\dagger = R_0$.

Set

$$R := C_0^{-1}R_0 = R_0C_0^{-1}, \quad H_3 := H_2C_0^{-1}, \quad C := H_1^{-1}H_3 = I - R.$$

Then $R^\dagger = (C_0^{-1})^\dagger R_0^\dagger = C_0^{-1}R_0 = R$ and since R is nilpotent we can proceed as in Lemma 7.8.1 and construct a unit $h \in E$ such that $h^*H_1h = H_3$ (where H_3 here plays the role of H_2 in that Lemma). Setting $f := h^{-1}$ and using that E is commutative, we obtain

$$f^*H_2f = f^*H_3C_0f = f^*H_3fC_0 = H_1C_0.$$

□

The previous proposition only gives a certain bound on the number of possible inequivalent compatible forms. For a more precise statement, we need to know more about the ground field $\mathbf{k}$. In the following we illustrate how this can work for the case when $\mathbf{k} = \mathbb{R}$. Since we are, after all, studying Hamiltonian vector fields in this section, we focus only on compatible skew-symmetric forms (i.e. symplectic forms).

PROPOSITION 6.1.23. *For the ground field, let* $\mathbf{k} = \mathbb{R}$. *Let* (V, X) *be an indecomposable self-dual endomorphism, with minimal polynomial* p^m, *where p is irreducible. Assume that* V *is even-dimensional. Up to equivalence, there exist precisely two compatible symplectic forms for* X.

PROOF. We work in coordinates with respect to a basis putting X into Mal'cev normal form A. We distinguish two cases, based on whether the self-dual polynomial $p(t)$ is of type I, i.e. $p(t) = t$, or of type II.

In the first case, when $p(t) = t$, the companion matrix Z of p is the 1×1 matrix (0), and so $F = \mathbf{k}(Z) = \mathbf{k}$. By Lemma 6.1.21, this means that any two compatible symplectic forms are equivalent, up to multiplication by a scalar in $\mathbb{R}$. Fix a compatible symplectic form H; then, for any other compatible symplectic form H', there exists an invertible scalar c such that H and cH' are equivalent. If c is positive, cH' is in fact equivalent to H via the isometry "multiplication by $\sqrt{c}$"; if c is negative, cH' is equivalent to $-H$. This shows that, up to equivalence, there is at most the compatible forms H and $-H$.

To prove that these two forms must be distinct, one may choose a specific H in coordinates (e.g. as constructed above), and then use the specific structure of elements of the endomorphism algebra of X in coordinates; see the proof of Theorem 8.3.12.

Now consider the second case, i.e. where p is of type II. Then p is necessarily of the form $p(t) = t^2 + a_0$ for some real scalar $a_0 > 0$. The roots of $p(t)$ in $\mathbb{C}$ are $\pm i\sqrt{a_0}$. The companion matrix Z of p is

$$\begin{bmatrix} 0 & -a_0 \\ 1 & 0 \end{bmatrix},$$

and the field $\mathbb{R}(Z)$ is a subfield of $\mathbb{R}^{2m \times 2m}$ which is isomorphic to $\mathbb{C}$: it consists of block diagonal matrices of the form $C\mathbb{1}$, where C is any 2×2 matrix of the form

$$\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$$

for $x, y \in \mathbb{R}$.

Again, we can choose a compatible symplectic form H for X in coordinates, e.g. of the kind in Proposition 6.1.17, and show that, up to equivalence, H and $-H$ are the only two such forms. For how one may spell this out, see Theorem 8.5.4. ☐

6.1.5. Classification of indecomposable linear Hamiltonian vector fields. We give the full classification for the case $\mathbf{k} = \mathbb{R}$. By Proposition 4.4.16, any given indecomposable linear Hamiltonian vectors field (V, X, ω) is of one of two possible types:

- Non-split: (V, X) is an indecomposable endomorphism (necessarily self-dual, since $\tilde{\omega} : (V, X) \to (V^*, -X^*)$).
- Split: $(V, X, \omega) \simeq (V_0 \oplus V_0^*, X_0 \oplus -X_0^*, \Omega)$, where Ω denotes the canonical hyperbolic symplectic structure on $V_0 \oplus V_0^*$, and (V_0, X_0) is an indecomposable endomorphism.

Recall that indecomposable endomorphisms are classified, up to isomorphism, by pairs (p, m), where p is an irreducible monic polynomial, and m is a positive integer (the indecomposable endomorphism corresponding to (p, m) has minimal polynomail p^m).

THEOREM 6.1.24. *The following is a complete classification of (non-zero) indecomposable linear Hamiltonian vector fields* (V, X, ω), *up to isometry.*

Non-split types:

(1) $(p, m, \pm)$, *where* $p(t) = t$, m *is even, and* $\pm$ *indicates that, up to isometry, there are two possible compatible symplectic forms. In other words, there are precisely two linear Hamiltonian vector fields of this type (up to isometry) for every even* $m \in \mathbb{Z}_+$.
(2) $(p, m, \pm)$, *where* $p(t) = t^2 + a_0$ *and* $a_0 > 0$. *Thus there are uncountably many indecomposable linear Hamiltonian vector fields of this type, two for every pair* (a_0, m), *where* $a_0 \in \mathbb{R}_+$ *and* $m \in \mathbb{Z}_+$.

Split types:

(1) (V_0, X_0) *is of type* (p, m), *where* $p(t) = t$ *and* m *is odd.*
(2) (V_0, X_0) *is of type* (p, m), *where* $p(t) = t + a_0$, *with* $a_0 \in \mathbb{R}_+$, *and* $m \in \mathbb{Z}_+$.
(3) (V_0, X_0) *is of type* (p, m), *where* $p(t) = t - 2xt + (x^2 + y^2)$, *with* $x \in \mathbb{R}_+, y \in \mathbb{R}_+$, *and* $m \in \mathbb{Z}_+$.

REMARK 6.1.25. Let (V, X, ω) be an indecomposable linear Hamiltonian vector field, and let σ be the spectrum of X in $\mathbb{C}$, i.e. its set of eigenvalues over $\mathbb{C}$. For the five types listed above, we have

Non-split types:

(1) $\sigma = \{0\}$
(2) $\sigma = \{\sqrt{-a_0}, -\sqrt{-a_0}\}$

Split types:

(1) $\sigma = \{0\}$
(2) $\sigma = \{a_0, -a_0\}$
(3) $\sigma = \{x + \sqrt{-1}y, x - \sqrt{-1}y, -x + \sqrt{-1}y, -x - \sqrt{-1}y\}$

PROOF OF THE THEOREM. As mentioned above, by Proposition 4.4.16 it is sufficient to classify the split and non-split types, respectively.

The list of indecomposable split types corresponds to the classification of non-selfdual indecomposable endomorphisms (and these correspond to those (p, m) for which p is *not* listed in (233)), together with the knowledge of which indecomposable endomorphisms are dual to which (this is given by Lemma 6.1.4). In the split types (2) and (3), respectively, we restrict to $a_0 \in \mathbb{R}_+$ and $x \in \mathbb{R}_+$ so that we only count *one* split indecomposable Hamiltonian vector field for every dual *pair* of indecomposable endomorphisms.

For the non-split types, the enumeration of the underlying indecomposable endomorphisms follows from Lemma 6.1.4 and from the existence result Proposition 6.1.14 that every such admits compatible symplectic forms when the ambient space is even dimensional. The uniqueness result Proposition 6.1.23 shows that there are in fact always precisely two compatible forms, up to isometry. □

REMARK 6.1.26. The classification given in Theorem 6.1.24 is consistent with the known classification of linear Hamiltonian vector fields as presented in [**LM74**] and [**Koc84**]. We indicate briefly how the types listed above correspond to the classifications in those papers.

In [**LM74**], the classification is split into four cases, and Case 3 is split into two subcases, involving two types of spaces, of the kind "U" or "Y". From Theorem 6.1.24, the non-split types (1) and (2) correspond to Case 3, Subcase "U", and Case 4, respectively, while the split types (1), (2), and (3) correspond to Case 3, Subcase "Y", Case 1, and Case 2, respectively.

In [**Koc84**], the classification is given twice, both times in terms of two different lists of normal forms. Each list is ordered into six types. We will compare with List II. The non-split types (1) respectively (2) of Theorem 6.1.24 correspond to the types (1) respectively (3) and (4) in List II of [**Koc84**]. The split types (1), (2) and (3) of Theorem 6.1.24 correspond to the types (2), (5), and (6), respectively, from [**Koc84**].

6.2. Linear symplectomorphisms

A linear symplectomorphism S on a symplectic vector space (V,ω) is a linear map $S : V \longrightarrow V$ such that

$$\begin{array}{ccc} V & \xrightarrow{\tilde{\omega}} & V^* \\ {\scriptstyle S}\downarrow & & \uparrow{\scriptstyle S^*} \\ V & \xrightarrow[\tilde{\omega}]{} & V^* \end{array} \tag{244}$$

commutes. In other words, S is an isometry of (V,ω). In particular, because ω is non-degenerate, S must be injective, and thus, since V is assumed to be finite-dimensional, S is invertible. Similar to the case of linear Hamiltonian vector fields, we will think in terms of (V,S) being an automorphism of V, together with a symplectic structure ω which is *compatible* with S in the sense that (244) commutes. This puts us in the situation of Example 3.2.6, with $\varepsilon = -1$. That is, a linear symplectomorphism (V,S,ω) is the same thing as a fixed point in the category $\mathrm{Aut}(\mathsf{vect}^-)$ whose objects are automorphisms (V,S) and which is equipped with the duality involution (δ,η) where

$$\delta(V,S) = (V^*, (S^{-1})^*) \tag{245}$$

and

$$\eta_V : V \longrightarrow V^{**}, v \longmapsto (\xi \mapsto -\xi(v)).$$

Here again, we have an obvious "inherited" notion of direct sum (and associated notions of indecomposability, etc.) making $\mathrm{Aut}(\mathsf{vect}^-)$ an additive category which satisfies the hypotheses of Section 4.4 and of Proposition 4.4.16 therein. Thus, we can follow the procedure outlined in Remark 6.1.1, and perform an analysis which is analogous to the one done above for linear Hamiltonian vector fields. We sketch this now, focusing again on the case then the ground field is $\mathbf{k} = \mathbb{R}$. The resulting full classification of indecomposable linear symplectomorphisms over $\mathbb{R}$ is given below in Section 6.2.3.

The first step of our procedure, namely a description of the indecomposable objects of $\mathrm{Aut}(\mathsf{vect}^-)$, is easy to give: up to isomorphism, indecomposable automorphisms (V,S) are simply indecomposable endomorphisms, with the additional condition that the endomorphism be invertible. In terms of the normal forms $(\mathbf{k}[t]/(p^m), M_{\underline{t}})$, invertibility corresponds to the requirement that the 0-th coefficient of the minimal polynomial p^m is non-zero. This is equivalent to requiring that that the 0-th coefficient of p itself is non-zero.

6.2.1. Duals. In order to identify which indecomposable automorphisms $(\mathbf{k}[t]/(p^m), M_{\underline{t}})$ are dual to which under our duality (245), we analyse the induced involution on minimal polynomials. It turns out to be the following one: given a monic polynomial with invertible 0-th coefficient

$$q(t) = a_k t^k + a_{k-1}t^{k-1} + \cdots + a_1 t + a_0 \qquad a_0 \neq 0,$$

we set

$$q^\circ(t) := t^k + \tfrac{a_1}{a_0}t^{k-1} + \cdots + \tfrac{a_{k-1}}{a_0}t + \tfrac{1}{a_0}.$$

If we think of $q(t)$ as encoded simply by its tuple of coefficients, then the above involution is

$$(1, a_{k-1}, .., a_1, a_0) \longmapsto \frac{1}{a_0}(a_0, a_1, .., a_{k-1}, 1).$$

It is straightforward to check that this involution is product preserving, i.e. $(q_1 q_2)^\circ = q_1^\circ q_2^\circ$.

LEMMA 6.2.1. *Let (V, S) be an automorphism. If $q(t)$ is the minimal polynomial of S, then $q^\circ(t)$ is the minimal polynomial of S^{-1}.*

PROOF. Suppose $q = (a_k, a_{k-1}, .., a_1, a_0)$. Since S is invertible, $a_0 \neq 0$. Let q' denote the minimal polynomial of S^{-1}.

By definition $q(S) = 0$ and $q^\circ(t) = \frac{1}{a_0} t^k q(t^{-1})$. Shus $q^\circ(S^{-1}) = 0$, and so $q'|q^\circ$. Shis implies that $\deg(q') \leq \deg(q^\circ) = k$.

On the other hand, replacing q by q' in the argument above, it follows that $q'^\circ((S^{-1})^{-1}) = q'^\circ(S) = 0$, and hence $q|q'^\circ$. So, $k = \deg(q) \leq \deg(q'^\circ) = \deg(q')$. We conclude that $\deg q' = k$ and $q' = q^\circ$. □

COROLLARY 6.2.2. *Let (V, S) be an indecomposable automrophism, with minimal polynomial $q = p^m$. Then its dual, $(V^*, (S^{-1})^*)$ is indecomposable, and has minimal polynomial $q^\circ = (p^\circ)^m$.*

COROLLARY 6.2.3. *The self-dual indecomposable automorphisms (V, S) are, up to isomorphism, precisely those with minimal polynomial $q = p^m$ such that*

$$p = p^\circ.$$

REMARK 6.2.4. Let $\mathbf{k} = \mathbb{R}$. Using the description (232) of irreducible polynomials over $\mathbb{R}$, it is easily seen that those with invertible 0-th coefficient and $p = p^\circ$ are precisely

$$p(t) = t \pm 1 \quad \text{and} \quad p(t) = t^2 - 2xt + (x^2 + y^2) \quad x \in \mathbb{R},\ y > 0,\ x^2 + y^2 = 1. \tag{246}$$

6.2.2. Compatible forms. Our goal in this section is to determine which self-dual automorphisms (V, S) admit compatible symplectic forms, and, for those that do, determine how many such exist up to isometry. Instead of performing an analysis analogous to the ones done in Section 6.1.3 and Section 6.1.4 for linear Hamiltonian vector fields, we will use here a trick involving Cayley transforms (this trick is used in [**LM74**]).

Let (V, X) be an endomorphism. If X is such that $X + 1$ is invertible, we define

$$\mathcal{T}_+ X := (X - 1)(X + 1)^{-1} \quad \text{and} \quad \mathcal{T}^- X := (1 - X)(1 + X)^{-1}. \tag{247}$$

Similarly, if X is such that $X - 1$ is invertible, we set

$$\mathcal{T}^+ X := (1 + X)(1 - X)^{-1} \quad \text{and} \quad \mathcal{T}_- X := (X + 1)(X - 1)^{-1}. \tag{248}$$

These are what we call Cayley transforms. Of course, $\mathcal{T}^- X$ is simply $-\mathcal{T}_+ X$ and $\mathcal{T}_- X = -\mathcal{T}^+ X$. However, we set this notation, since $\mathcal{T}_+ X$ and $\mathcal{T}_- X$ will play analogous roles.

REMARK 6.2.5. $X + 1$ invertible $\Leftrightarrow$ -1 is not an eigenvalue of X. Similarly, $X - 1$ invertible $\Leftrightarrow$ 1 is not an eigenvalue of X.

LEMMA 6.2.6. *Suppose $X + 1$ is invertible, and suppose λ is an eigenvalue of X. Then $\lambda + 1 \neq 0$ and $\frac{\lambda+1}{\lambda-1}$ is an eigenvalue of $\mathcal{T}_+ X$.*

The analogous statements hold also for the other Cayley transforms above.

PROOF. Let v be an eigenvector of X for the eigenvalue λ (in particular $v \neq 0$). Then $(X+1)v = (\lambda+1)v$; if $\lambda+1$ were zero, then $X+1$ would have non-zero kernel, a contradiction.

Now note that

$$\mathcal{T}_+X(\lambda+1)v = (X-1)(X+1)^{-1}(X+1)v = (X-1)v = (\lambda-1)v,$$

so

$$\mathcal{T}_+Xv = \tfrac{\lambda+1}{\lambda-1}v.$$

□

LEMMA 6.2.7. *If $X+1$ is invertible, then $\mathcal{T}_+X-1$ is invertible and $\mathcal{T}^+\mathcal{T}_+X = X$. An analogous statement holds for $\mathcal{T}_+\mathcal{T}^+X = X$ (when defined), and similarly for $\mathcal{T}_-$ and $\mathcal{T}^-$.*

PROOF. First we show that $\mathcal{T}_+X-1$ is invertible when $X+1$ is. Suppose $\mathcal{T}_+X-1$ is not invertible, i.e. $\mathcal{T}_+X$ has 1 as an eigenvalue. Let $v \neq 0$ be an associated eigenvector. Then $\mathcal{T}_+Xv = v$ implies $(X-1)v = (X+1)v$, which implies $0 = 2v$, which (since $\mathrm{char}(\mathbf{k}) \neq 2$) is only possible if $v = 0$, but that is contradiction.

Now we simply compute:

$$\mathcal{T}^+\mathcal{T}_+X = \mathcal{T}^+(X-1)(X+1)^{-1} = [1+(X-1)(X+1)^{-1}][1-(X-1)(X+1)^{-1}]^{-1}$$
$$(X+1)^{-1}[(X+1)+(X-1)][(X+1)-(X-1)]^{-1}(X+1) = [2X][2]^{-1} = X.$$

□

LEMMA 6.2.8. *Let (V,X) be an endomorphism, and suppose (V,X) is decomposable. Let $\mathcal{T}$ be any of the four Cayley transforms above. If defined, then $(V,\mathcal{T}X)$ is also decomposable.*

PROOF. Decompositions of (V,X) correspond to non-trivial idempotents of (V,X); these are precisely non-trivial idempotents of V which commute with X. Such an idempotent will then also commute with $\mathcal{T}X$, giving non-trivial decompositions of these endomorphisms as well. □

COROLLARY 6.2.9. *Let (V,X) be an indecomposable endomorphism, and let $\mathcal{T}$ be one of the four Cayley transforms. If $\mathcal{T}X$ is defined, then $(V,\mathcal{T}X)$ is also indecomposable.*

LEMMA 6.2.10. *Let (V,S) be an automorphism, and ω a symplectic structure on V.*

(1) *If $X+1$ is invertible, then (V,S,ω) is a linear symplectomorphism if, and only if, $(V,\mathcal{T}_+X,\omega)$ is a linear Hamiltonian vector field.*
(2) *If $X-1$ is invertible, then (V,S,ω) is a linear symplectomorphism if, and only if, $(V,\mathcal{T}_-X,\omega)$ is a linear Hamiltonian vector field.*

PROOF. The proof is a straightforward calculation; see [**LM74**], page 236. □

Now let us assume $\mathbf{k} = \mathbb{R}$ and consider a self-dual indecomposable automorphism (V,S) on even-dimensional V. We know that S has minimal polynomial of the form $q = p^m$, where p is a monic irreducible which is one of the polynomials (246). We consider three cases in turn.

<u>$p(t) = t-1$:</u>

In this case, S has 1 as its unique eigenvalue, and so $S+1$ is invertible. We apply $\mathcal{T}_+$, which gives an indecomposable endomorphism $(V,\mathcal{T}_+S)$ having unique eigenvalue $(1-$

$1)(1+1)^{-1} = 0$. Since V is even dimensional, this is of non-split type "(1)" as in Theorem 6.1.24 and Remark 6.2.12. By Lemma 6.2.10 compatible forms for (V, S) are identical with compatible forms for $(V, \mathcal{T}_+ S)$.

$p(t) = t + 1$:

Here S has -1 as its unique eigenvalue, so $S - 1$ is invertible and we use $\mathcal{T}_-$. The indecomposable endomorphism $(V, \mathcal{T}_- S)$ again has $(-1+1)(-1-1)^{-1} = 0$ as its unique eigenvalue. And again, since V is even dimensional, this is of non-split type "(1)". By Lemma 6.2.10 compatible forms for (V, S) are identical with compatible forms for $(V, \mathcal{T}_+ S)$.

$p(t) = t^2 - 2xt + 1$, where $x \in (-1, 1)$:

Now S has no eigenvalue in $\mathbb{R}$, and eigenvalues away from ± 1 in $\mathbb{C}$, so both $S + 1$ and $S - 1$ are invertible over both $\mathbb{R}$ and $\mathbb{C}$. We apply $\mathcal{T}_+$, obtaining an indecomposable endomorphism $(V, \mathcal{T}_+ S)$. If $\lambda = x \pm \sqrt{-1}y$ are the complex eigenvalues of S, then $(\lambda - 1)(\lambda + 1)^{-1} = \pm\sqrt{-1}y(1 + x)^{-1}$. This is of non-split type "(2)", c.f. Remark 6.2.12. By Lemma 6.2.10 compatible forms for (V, S) are identical with compatible forms for $(V, \mathcal{T}_+ S)$.

6.2.3. Classification of indecomposable linear symplectomorphisms. We know that every indecomposable linear symplectomorphism (V, S, ω) is of one of two types:

- Non-split: (V, S) is an indecomposable automorphism (necessarily self-dual, since $\tilde{\omega} : (V, X) \to (V^*, -X^*)$).
- Split: $(V, S, \omega) \simeq (V_0 \oplus V_0^*, X_0 \oplus -X_0^*, \Omega)$, where Ω denotes the canonical hyberbolic sympeletic structure on $V_0 \oplus V_0^*$, and (V_0, X_0) is an indecomposable automorphism.

The following classification is a consequence of Theorem 6.1.24 and the analysis done in the previous two sections.

Theorem 6.2.11. *A complete classification of (non-zero) indecomposable linear symplectomorphisms (V, S, ω), up to isometry, is as follows.*

Non-split types:

(1) *$(p, m, \pm)$, where $p(t) = t - 1$, m is even, and $\pm$ indicates that, up to isometry, there are two possible compatible symplectic forms. In other words, there are precisely two linear symplectomorphisms of this type (up to isometry) for every even $m \in \mathbb{Z}_+$.*

(2) *$(p, m, \pm)$, where $p(t) = t + 1$, m is even, and $\pm$ indicates that, up to isometry, there are two possible compatible symplectic forms.*

(3) *$(p, m, \pm)$, where $p(t) = t^2 - 2xt + 1$. There are uncountably many indecomposable linear symplectomorphisms of this type, two for every pair (x, m), where $x \in (-1, 1) \subseteq \mathbb{R}$ and $m \in \mathbb{Z}_+$.*

Split types:

(1) *(V_0, X_0) is of type (p, m), where $p(t) = t - 1$ and m is odd.*

(2) *(V_0, X_0) is of type (p, m), where $p(t) = t + 1$ and m is odd.*

(3) *(V_0, X_0) is of type (p, m), where $p(t) = t + a_0$, with $a_0 \in (-1, 1)\backslash\{0\}$, and $m \in \mathbb{Z}_+$.*

(4) *(V_0, X_0) is of type (p, m), where $p(t) = t - 2xt + (x^2 + y^2)$, with $0 < y, 0 < x^2 + y^2 < 1$, and $m \in \mathbb{Z}_+$.*

REMARK 6.2.12. Let (V, S, ω) be an indecomposable linear symplectomorphism, and let σ be the spectrum of S in $\mathbb{C}$. For the five types listed above, we have

Non-split types:

(1) $\sigma = \{1\}$
(2) $\sigma = \{-1\}$
(3) $\sigma = \{\lambda, \overline{\lambda}\}$ for $\lambda \in S^1 \backslash \{-1, 1\}$.

Split types:

(1) $\sigma = \{1\}$
(2) $\sigma = \{-1\}$
(3) $\sigma = \{\lambda, \lambda^{-1}\}$ for $\lambda \in \mathbb{R} \backslash \{0\}$.
(4) $\sigma = \{\lambda, \overline{\lambda}, \lambda^{-1}, \overline{\lambda}^{-1}\}$ for $\lambda \in \mathbb{C} \backslash (S^1 \cup \mathbb{R})$.

CHAPTER 7

Symplectic poset representations

Most of the contents of this chapter, as well as the next one, are taken directly from the paper [**HLW19**], which uses symplectic poset representations to classifying triples of isotropic subspaces in symplectic vector spaces. The main purpose of this short introduction is to connect the material from [**HLW19**], which is written in a non-categorical style, to the general framework of duality involutions in additive categories, as covered in Part 1. In other words, we have reproduced [**HLW19**] in a way which preserves its semantic and logical integrity, and we indicate briefly here beforehand the connections to the broader picture.

One reason for studying poset representations is that they are a tool for studying various classification problems of linear algebra within a single formalism. If one is dealing with questions that involve symmetric or skew-symmetric bilinear forms, it is possible to study these using a variant of poset representations which is also equipped with such extra geometric structures. Our focus is on classification problems in symplectic linear algebra, and the formalism which we use to study these is that of "symplectic poset representations".

Although the following definitions will also be introduced again in detail below, let us briefly explain the basic set-up. Given a finite poset P, a linear representation of P on a (finite-dimensional) vector space V is an order-preserving map from P to the poset $\mathrm{Sub}(V)$ of linear subspaces of V, ordered by inclusion. Poset representations may be organized into the structure of an additive category $\mathrm{Rep}(P)$. Given representations $\psi : P \to \mathrm{Sub}V$ and $\psi' : P \to \mathrm{Sub}V'$, a morphism $\psi \to \psi'$ is a linear map $f : V \to V'$ such that $f(\psi(x)) \subseteq \psi'(x)$ for all $x \in P$. The zero object is the (unique) representation on the zero vector space, and the biproduct of the representations ψ and ψ' is defined on $V \oplus V'$ by

$$(\psi \oplus \psi')(x) := \psi(x) \oplus \psi'(x), \qquad x \in P.$$

In order to encode subspaces of *symplectic* vector spaces, we use duality involutions, and capture the symplectic structure as part of a fixed point structure. Specifically, we assume that our poset P is now equipped with an order reversing involution, which we denote by $(-)^{\perp}$, and we define a duality in involution on $\mathrm{Rep}(P)$ by defining the dual of a representation ψ on V as a representation on V^* given by

$$\psi^*(x) := \psi(x^{\perp})^{\circ}, \qquad x \in P,$$

where $(-)^{\circ}$ denotes the operation of taking the annihilator. The dual of a morphism of poset representations is given by the adjoint linear map. This defines an additive duality involution on $\mathrm{Rep}(P)$, with the unit given in components by

$$\eta_V : V \longrightarrow V^{**}, \ v \longmapsto (\xi \mapsto -\xi(v)).$$

Fixed points of this duality involution are what we call symplectic poset representations. Such a fixed point encodes precisely the data of a symplectic vector space (V, ω) together with a poset representation $\psi : P \to V$ which satisfies the relation

$$\psi(x^{\perp}) = \psi(x)^{\perp},$$

where the latter "$(-)^{\perp}$" denotes the operation of taking the symplectic orthogonal subspace with respect to ω.

The Krull-Schmidt theorem holds in the category $\mathrm{Rep}(P)$ (the properties of Definition 4.3.1 are satisfied), and given a classification of indecomposables in category $\mathrm{Rep}(P)$, we may follow the strategy outlined in Remark 4.4.17 in order to classify indecomposable symplectic poset representations. Indeed, we can use Proposition 4.4.16 since although $\mathrm{Rep}(P)$ does *not* have all cokernels (so it is not abelian), $\mathrm{Rep}(P)$ does have all kernels, so in particular it is idempotent complete (by Lemma 1.5.24). Furthermore, the Fitting lemma holds in $\mathrm{Rep}(P)$. We note the that, below, the hyperbolization construction of Section 4.2 is used under the name of "symplectization".

The remainder of the present chapter is devoted to general aspects of symplectic poset representations, while Chapter 8 discusses in detail the classification of indecomposable triples of isotropic subspaces, which corresponds to classifying indecomposable symplectic poset representation of a certain six-element poset equipped with a specific order-reversing involution.

In the following, we work with a fixed ground field, which we assume to be perfect (e.g. of characteristic zero, finite, or algebraically closed) and not of characteristic 2; otherwise we usually leave the field unspecified. All vector spaces are assumed to be finite-dimensional.

7.1. Basic notions

A pair of subspaces (A, B) in a vector space V (without further structure) is completely determined up to isomorphism by four invariants: the dimensions of V, A, B, and $A \cap B$. For a triple $(A, B, C) \subseteq V$ of subspaces, one needs to know, in addition to the dimensions of V, A, B, C, all pairwise intersections, and the triple intersection, the dimension of one more space, such as $(A+B) \cap C$, giving a total of nine invariants. (For instance, this ninth invariant is needed to distinguish the two arrangements of three distinct lines in 3-space, which may or may not be coplanar.) At this point, we have effectively introduced the lattice $\langle A, B, C\rangle$ generated by A, B, and C as a sublattice of the lattice $\Sigma(V)$ of subspaces in V, i.e. the subspaces generated by A, B, and C under iterations of the operations of sum and intersection. The study of such structures dates back to Dedekind and Thrall, immanent in early results of representation theory and leading into abstract lattice theory. The present work is mainly oriented around posets, with $\Sigma(V)$ being a poset with respect to inclusion, and we will not deal with abstract lattice theory. Nevertheless, the lattice structure of $\Sigma(V)$ will play an important role, in particular in calculations.

A non-degenerate antisymmetric bilinear, i.e. **symplectic**, form ω on V produces additional structure.

First of all, there is a naturally associated linear map $\tilde{\omega}$ from V to V^* defined by $\tilde{\omega}(v)(w) = \omega(v, w)$. Non-degeneracy means that $\tilde{\omega}$ is an isomorphism.

Next, there is a natural order-reversing involution $A \mapsto A^{\perp}$ on $\Sigma(V)$, where $A^{\perp}$, the **symplectic orthogonal** of A, is the subspace $\{v \in V \mid \forall w \in A, \omega(v, w) = 0\}$. This involution is related to another order-reversing operation, namely the one that maps a subspace $A \subseteq V$ to its annihilator $A^{\circ} = \{\xi \in V^* \mid \xi(v) = 0 \ \forall v \in A\}$ in V^*. The following result is easy to verify; we state it as a lemma to refer to later.

Lemma 7.1.1. *For any subspace A in a symplectic space (V,ω), $\tilde{\omega}(A^\perp) \subseteq V^*$ is the annihilator A° of A.*

Using the involution $\perp$, we may define several special types of subspaces. There are the **isotropic** subspaces, for which $A \subseteq A^\perp$, and the **coisotropic** subspaces, for which $A^\perp \subseteq A$. Subspaces which are both isotropic and coisotropic, i.e. fixed points of the orthogonality involution, are called **lagrangian**. Subspaces for which $A \cap A^\perp = \{0\}$ are called **symplectic**; the restriction of the symplectic form to such a subspace is non-degenerate, and hence again a symplectic form. For such spaces, the order-reversing property implies that this condition is equivalent to $A + A^\perp = V$, so that we can also write the condition as $V = A \oplus A^\perp$. Involutivity implies that $A^\perp$ is symplectic as well, giving a **symplectic direct sum decomposition** of V. Such symplectic direct sums can also be built from the external direct sum of symplectic spaces by equipping the sum with the direct sum form, which is again symplectic. The aim in the present work is to study the decomposition of (co)isotropic triples with respect to symplectic direct sums (not just of two summands).

Given a symplectic direct sum decomposition $V = A_1 \oplus A_2$, we can define in a purely lattice-theoretic way the involution on $\Sigma(A_1)$ associated with the restricted symplectic structure. Let C_1 be any subspace of A_1. Then $C_1^\perp$ contains A_2, and the modular law[1] implies that $C_1^\perp = (C_1^\perp \cap A_1) \oplus A_2$. So we may define the operation $(-)^{\perp_1}$ by setting $C_1^{\perp_1} := C_1^\perp \cap A_1$. It is clearly order-reversing, and it is easy to check that it is involutive. Of course, we can do the same thing in A_2. We see from here that for subspaces $C_j \subseteq A_j$, the direct sum $C_1 \oplus C_2$ is (co)isotropic or symplectic if and only if C_1 and C_2 are. Similar statements to all of those above hold for orthogonal decompositions with any number of summands.

A **linear representation**[2] of a partially ordered set, or poset, P in a vector space V is an order preserving map ψ from P to the poset $\Sigma(V)$. The **dimension vector** of ψ is the function which assigns to each $p \in P$ the dimension of $\psi(p)$. We will also add $\dim V$ as a first component to the dimension vector. A **morphism** $f : \psi \to \psi'$ between representations ψ in V and ψ' in V' of the same poset P is a linear map $f : V \to V'$ such that $f(\psi(p)) \subseteq \psi'(p)$ for all $p \in V$; it is an **isomorphism** if f is bijective and $f(\psi(p)) = \psi'(p)$ for all $p \in P$.

The term **poset with involution** will denote a poset P equipped with an order-reversing involution $\perp: P \to P$. The relevance of poset representations and involutions arises when we observe that each isotropic subspace A in a symplectic space V is naturally associated with the subspace $A^\perp$ which contains it and hence with the pair $(A, A^\perp)$. Conversely, if (A, B) is any pair of subspaces for which $A \subseteq B$ and $A^\perp = B$, then A is isotropic and B is its coisotropic orthogonal. Thus we see that isotropic (or equivalently, coisotropic) subspaces in V may be identified with involution-preserving representations into $\Sigma(V)$ of a poset consisting of two elements in linear order, equipped with the order-reversing involution which exchanges the two elements. We will denote this poset by **2**.

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[1]The modular law states that, for subspaces A, B, C of a given vector space, if $B \subseteq A$, then $A \cap (B + C) = B + A \cap C$.

[2]We use the word "linear" here simply to distinguish this notion of representation from the notion of symplectic poset representation, which we define below.

Definition 7.1.2. *Let* $(P, \perp)$ *be a poset with involution,* (V, ω) *a symplectic vector space, and* $(\Sigma(V), \perp)$ *the poset of its subspaces equipped with the involution induced by* ω.[3] *A* **symplectic representation** *of* $(P, \perp)$ *in* V *is an order-preserving map* $\varphi : P \to \Sigma(V)$ *such that*

$$\varphi(p^{\perp}) = \varphi(p)^{\perp} \quad \forall p \in P.$$

If φ *is such a representation, forgetting the involutions gives us simply a linear representation of* P *in the vector space* V*, which we denote by* $\hat{\varphi}$ *and call the* ***underlying*** *linear representation.*

Given another symplectic representation φ' *in* (V', ω') *of the same poset with involution* $(P, \perp)$*, an* **isomorphism** *from* φ *to* φ' *is an isomorphism of linear poset representations* $\hat{\varphi} \to \hat{\varphi}'$ *which is also an isometry* $(V, \omega) \to (V', \omega')$.

Now, for any k, the k-tuples of isotropic-coisotropic pairs in (V, ω) are simply the symplectic representations in (V, ω) of the partially ordered set $k \cdot \mathbf{2} := \mathbf{2} + \mathbf{2} + \cdots + \mathbf{2}$ (k times) consisting of k copies of $\mathbf{2}$ which are independent in the sense that $a \leq b$ only when a and b belong to the same copy, with the involution interchanging the elements of each copy of $\mathbf{2}$. In the present work, we derive for $k = 0, 1, 2, 3$, up to direct decomposition, the classification of isotropic k-tuples from the classification of linear representations of the poset $k \cdot \mathbf{2}$; these essentially correspond to the representations of a quiver associated with the Dynkin diagrams A_1, A_3, and A_5, and the extended Dynkin diagram $\tilde{E}_6$, respectively. The first three cases are easy, with only finitely many indecomposables up to isomorphism, while the latter case is more involved – in particular there are infinitely many indecomposables, some appearing in 1-parameter families.

Our general strategy for classifying isotropic triples will involve three basic operations related to symplectic representations. First of all, given a symplectic representation φ of $(P, \perp)$ in (V, ω), one may ignore the involution and symplectic structure to obtain a linear representation $\hat{\varphi}$ of P in V. Second, starting with a linear representation ψ of $(P, \perp)$ in V, one can ask whether there exists a symplectic form on V such that ψ is in fact a symplectic representation in (V, ω); in this case ω is a **compatible** symplectic form. Third, one can build a symplectic representation out of each linear one by a "doubling" construction called **symplectification**, which we define in subsection 7.6 below.

Remark 7.1.3. These operations are close to ones in symplectic geometry. The first one corresponds to forgetting the symplectic structure on a symplectic manifold, the second is analogous to asking whether a given manifold admits a symplectic structure, while the third is similar to the cotangent bundle construction.

7.2. Decompositions of linear representations

We fix a poset P. Given linear poset representations ψ and ψ', on V and V' respectively, their (external) **direct sum** is the poset representation on $V \oplus V'$ defined by

$$(\psi \oplus \psi')(x) = \psi(x) \oplus \psi'(x) \qquad \forall x \in P.$$

A **subrepresentation** of a linear representation ψ on V is a representation ψ' on a subspace $U \subseteq V$ such that $\psi'(x) \subseteq \psi(x)$ $\forall x \in P$. Given subrepresentations ψ' and ψ'' of ψ, on U' and U'' respectively, we say they form an (internal) **direct sum decomposition** of ψ if $V = U' \oplus U''$ and

$$\psi(x) = \psi'(x) \oplus \psi''(x) \qquad \forall x \in P.$$

[3] We use the same symbol to denote two different involutions – the one on P and the one on $\Sigma(V)$.

In this case, $\psi'(x) = \psi(x) \cap U'$ for all $x \in P$, and similarly for ψ''.

We note that, given a subrepresentation ψ' of ψ, there may not exist a subrepresentation ψ'' such that $\psi = \psi' \oplus \psi''$. For an example, consider $P = \{x_1, x_2, x_3\}$ endowed with the empty partial order, i.e. there are no order relations between the elements. Let ψ be a representation of P on a two-dimensional space V such that the subspaces $\psi(x_i)$ are three independent lines, and let ψ' be the subrepresentation on $U' := \psi(x_1)$ such that $\psi'(x_1) = \psi(x_1)$ and $\psi'(x_2) = \psi'(x_3) = 0$. Suppose there existed a subrepresentation ψ'' such that $\psi = \psi' \oplus \psi''$. Then ψ'' would be need to be defined on a subspace U'' such that $V = U' \oplus U''$; thus U'' would be a line in V. By the requirement that $\psi(x_i) = \psi'(x_i) \oplus \psi''(x_i)$, it follows that necessarily

$$\psi(x_2) = 0 \oplus \psi''(x_2) \quad \text{and} \quad \psi(x_3) = 0 \oplus \psi''(x_3).$$

But since the subspaces $\psi''(x_2)$ and $\psi''(x_3)$ must be contained in the line U'', this would imply that $\psi''(x_2) = \psi''(x_3) = U''$, a contradiction to the fact that $\psi''(x_2)$ and $\psi''(x_3)$ are independent.

For any vector space V, idempotents π in the algebra $\mathrm{End}(V)$ of its endomorphisms correspond to direct sum decompositions $V = A \oplus B$ of V, where $A = \mathrm{Im}\,\pi$ and $B = \mathrm{Ker}\,\pi = \mathrm{Im}\,(1-\pi)$. (The zero and the identity endomorphisms are denoted by 0 and 1, respectively.) The same is true for the endomorphism algebra $\mathrm{End}(V, \psi)$ of a linear representation ψ in V of a fixed partially ordered set P. Recall that an endomorphism of a linear representation ψ is a linear map $f : V \to V$ such that $f(\psi(p)) \subseteq \psi(p)$ for all $p \in P$. It is easy to check that $\mathrm{End}(V, \psi)$ is a unital subalgebra of $\mathrm{End}(V)$. Since the ground field embeds into $\mathrm{End}(V, \psi)$ via its action on the algebra unit, we sometimes refer simply to the *ring* of endomorphisms of ψ.

The basic theory of poset representations parallels that of modules of finite composition length as presented, for example, in [**Lam66**, §1.4]. We nevertheless give some outlines of proofs, for the convenience of the reader.

We call a subring E of some $\mathrm{End}(V)$ **local** if each of its elements is either invertible or nilpotent. It follows that, for any $f \in E$, either f or $1 - f$ is invertible; namely, if f is not invertible, then $f^n = 0$ for some n and $1 - f$ has inverse $\sum_{i=0}^{n} f^i$. More generally, if $g = \sum_{i=1}^{m} f_i$ is invertible then so is at least one of the f_i. The nilpotent elements form an ideal $\mathrm{rad}\,E$, the **radical** of E. (Namely, given f and g, if one of them is nilpotent and if fg were to be invertible, then f would be surjective and g injective, so actually both would be invertible. And if both f and g are nilpotent then $f + g$ cannot be invertible.) It follows that $E/\mathrm{rad}\,E$ is a division ring and $\mathrm{rad}\,E$ the unique maximal ideal.

The following result is a version of Fitting's Lemma.

LEMMA 7.2.1. *For a linear poset representation ψ in V the following are equivalent*

(1) *ψ is indecomposable.*
(2) *$\mathrm{End}(V, \psi)$ is local.*
(3) *$\mathrm{End}(V, \psi)$ has only the trivial idempotents 0 and 1.*

PROOF. Clearly, 2) ⇒ 3) ⇒ 1). Now, assume ψ indecomposable. The rank of f^j is a non-increasing and non-negative function of $j = 1, 2, 3, \cdots$, so it stabilizes after finitely many steps, say d steps. Then $f^j(\mathrm{Im}\, f^d) = \mathrm{Im}\, f^{d+j} = \mathrm{Im}\, f^d$ for all negative j; Call this image space I. It follows that the nondecreasing sequence of kernels of these powers also satisfies $N := \mathrm{Ker}\, f^{d+j} = \mathrm{Ker}\, f^d$; call this null space N. If $v \in I \cap N$, then $v = f^d(w)$ for some w, and $0 = f^d v$. Hence, $f^{2d}(w) = 0$, so $v = f^d(w)$ must be zero as well. So

$I \cap N = \{0\}$. By dimension counting, we have $V = I \oplus N$, where I and N are both f-invariant, $f|_I$ is invertible (since it is surjective), and f_N is nilpotent.

If K is any f-invariant subspace, then $f^k(K) \subseteq K$ is the projection $\pi_I(K)$ of K on I. Then the other projection $\pi_N(K) = (1 - \pi_I)(K)$ is contained in K as well. This shows that any f-invariant subspace is the direct sum of its components in I and N. It follows that, if f leaves invariant an indecomposable family of subspaces, either I or N must be zero, and f is either nilpotent or invertible. □

REMARK 7.2.2. Even without the indecomposability assumption, the decomposition into invertible and nilpotent parts is unique: for any such decomposition $f_I \oplus f_N$, I and N must be the image and kernel respectively of all sufficiently large powers of f.

We now state the Krull-Remak-Schmidt theorem in the form that we will need.

THEOREM 7.2.3. *Let $\psi = \psi_1 \oplus \psi_2 \oplus \cdots \oplus \psi_n$ and let $\psi' = \psi'_1 \oplus \psi'_2 \oplus \cdots \oplus \psi'_{n'}$ be direct sum decompositions of isomorphic linear poset representations, e.g., of the same representation, into indecomposable summands. (Such decompositions always exist in finite dimensions.) Then $n = n'$, and the two decompositions are the same up to isomorphism and permutation of the summands.*

PROOF. We may assume $n' \geq n$, and we denote the underlying vector spaces as $V_1 \oplus \cdots \oplus V_n$ and $V'_1 \oplus \cdots \oplus V'_{n'}$. First, consider $n = n' = 2$ and an isomorphism $f : \psi \to \psi'$ such that $f(v_1, 0) = (g(v_1), h(v_1))$ with morphisms $g, h : \psi_1 \to \psi'_1$ where g is an isomorphism. We claim that ψ_2 and ψ'_2 are isomorphic, too. To prove this, we may assume $h = 0$: replace f by f' where $f'(v_1, v_2) = (w_1, w_2 - hg^{-1}(w_1))$ if $f(v_1, v_2) = (w_1, w_2)$. Then $\psi_2 \cong \psi/(\psi_1 \oplus 0) \cong (f\psi)/(g\psi_1 \oplus 0) = \psi'/(\psi'_1 \oplus 0) \cong \psi'_2$.

In general, let $f : \psi \to \psi'$ be a given isomorphism and consider the canonical embeddings $\varepsilon_i, \varepsilon'_i$ and projections π_i, π'_i given by the decompositions. Put $g_i = \pi'_1 \circ f \circ \varepsilon_i$ and $h_i = \pi_i \circ f^{-1} \circ \varepsilon'_1$. Then $\sum_{i=1}^n g_i \circ h_i = 1 \in \mathrm{End}(V'_1, \psi'_1)$. Since this ring is local, one of the summands is invertible, say the first. Then $h_1 \circ g_1 \in \mathrm{End}(V_1, \psi_1)$ is not nilpotent, whence invertible. Thus, $g_1 : \psi_1 \to \psi'_1$ is an isomorphism. Clearly, $f(v_1, 0, \ldots, 0) = (g_1(v_1), w_2, \ldots)$ and it follows, by the special case, that $\psi_2 \oplus \cdots \oplus \psi_n$ is isomorphic to $\psi'_2 \oplus \cdots \oplus \psi'_{n'}$. We repeat the argument until only ψ_n is left on one side. Since ψ_n is indecomposable one has $n = n'$ whence $\psi_n \cong \psi'_n$.

□

7.3. Endomorphisms

Poset representations of particular interest are ones which can be built from a vector space U and a linear map $\eta : U \to U$. We will will refer to such a couple (U, η) simply as an **endomorphism** when no confusion with other notions is to be feared. From such an endomorphism one can build a quadruple of subspaces, which is thus aa poset representation $(V; U_1, U_2, U_3, U_4)$, i.e. a representation of the poset **1+1+1+1**: define

$$\begin{aligned} V &= U \times U \\ U_1 &= U \times 0 \\ U_2 &= 0 \times U \\ U_3 &= \{(x, -x) \mid x \in V\} \\ U_4 &= \{(x, -\eta(x)) \mid x \in V\}. \end{aligned} \tag{249}$$

We will call this the **quadruple associated to** (U, η). Note that U_4 and U_3 are the negative graphs, respectively, of η and the identity map on U. It is straightforward to check that any endomorphism of the poset representation (249) is of the form $f \times f$, where $f : U \to U$ is such that

$$\eta \circ f = f \circ \eta. \tag{250}$$

The collection of linear maps $f : U \to U$ satisfying (250) form an algebra $\mathrm{End}(U, \eta)$, the **endomorphism algebra of** (U, η). More generally, a **morphism** of endomorphisms $(U, \eta) \to (U', \eta')$ is a linear map $f : U \to U'$ satisfying $\eta' \circ f = f \circ \eta$; such a map is an **isomorphism** if, additionally, f is invertible as a linear map.

LEMMA 7.3.1. *A poset representation* $(V; U_1, U_2, U_3, U_4)$ *is isomorphic to one of the form* (249) *for some* (U, η) *if and only if*

$$U_i \oplus U_j = V \text{ for } i < j \leq 3 \quad \text{and} \quad U_2 \oplus U_4 = V. \tag{251}$$

PROOF. It is easily seen that, given (U, η), the associated quadruple satisfies (251). For the converse, suppose $(V; U_1, U_2, U_3, U_4)$ is a quadruple satisfying (251). First, note that the conditions (251) imply that all of the subspaces U_i have the same dimension. Choose any vector space U having the same dimension as the U_i and choose linear isomorphisms $\varphi_1 : U_1 \to U$ and $\varphi_2 : U_2 \to U$. Next, note that since U_3 is independent of both U_1 and U_2, there exists an isomorphism $\varphi_3 : U_1 \to U_2$ such that $U_3 = \{x + \varphi_3 x \mid x \in U_1\} \subseteq U_1 \oplus U_2$. Finally, using this data we construct the following map

$$\varphi : V = U_1 \oplus U_2 \longrightarrow U \times U, \ x + y \longmapsto (\varphi_1 x, -\varphi_1 \varphi_3^{-1} y).$$

Clearly φ is a linear isomorphism and maps U_1 to $U \times 0$ and U_2 to $0 \times U$. Furthermore, given $x + \varphi_3 x \in U_3$, its image under φ is $(\varphi_1 x, -\varphi_1 x)$, as desired. Since U_4 is independent of U_2, it is the graph of a linear map $\varphi_4 : U_1 \to U_2$ (though this map may not be an isomorphism). Now given $x + \varphi_4 x \in U_4 \subseteq U_1 \oplus U_2$, we have

$$\varphi(x + \varphi_4 x) = (\varphi_1 x, -\varphi_1 \varphi_3^{-1} \varphi_4 x) = (\varphi_1 x, -\varphi_1 \varphi_3^{-1} \varphi_4 \varphi_1^{-1} \varphi_1 x).$$

Setting $\eta = \varphi_1 \varphi_3^{-1} \varphi_4 \varphi_1^{-1}$, we have a quadruple of the form (249) which is isomorphic to $(V; U_1, U_2, U_3, U_4)$. □

An endomorphism (U, η) is **decomposable** if there exist non-zero subspaces $U_1, U_2 \subseteq U$ which are invariant under η and form a decomposition $U = U_1 \oplus U_2$. As in the case of poset representations, such decompositions correspond to idempotents in the endomorphism algebra of (U, η). For future reference we state:

LEMMA 7.3.2. *The endomorphism algebra of* (U, η) *is isomorphic to the endomorphism algebra of the associated quadruple* (249)*. In particular,* (U, η) *is indecomposable if and only if its endomorphism algebra is local.*

PROOF. An isomorphism is given by mapping an endomorphism f of (U, η) to the endomorphism $f \times f$ of (249). Its inverse is "restriction to U_1".

The characterization of indecomposabilty follows from Lemma 7.2.1. □

Another point of view is that an endomorphism (U, η) defines a $\mathbf{k}[x]$-module $U_{\mathbf{k}[\eta]}$, where the action of x on $U_{\mathbf{k}[\eta]}$ is defined by the action of η on U, i.e. $(\sum_i \lambda_i x^i) \cdot u := (\sum_i \lambda_i \eta^i)(u)$ for $u \in U$, $\lambda_i \in \mathbf{k}$. In this case, morphisms $f : (U, \eta) \to (U', \eta')$ are the same as module homomorphisms $f : U_{\mathbf{k}[\eta]} \to U'_{\mathbf{k}[\eta']}$ and decompositions of (U, η) correspond to direct sum decompositions of $U_{\mathbf{k}[\eta]}$.

LEMMA 7.3.3. *If η is an indecomposable endomorphism of U with an eigenvalue λ in the base field* $\mathbf{k}$*, then there is a basis such that η is described by a single λ-Jordan block. With respect to such a basis, the members of the endomorphism algebra $E = End(U, \eta)$ are given by the matrices $\sum_{i=0}^{d-1} a_i N^i$ where $d = \dim U$ and N is the nilpotent matrix with $n_{i,i+1} = 1$ for $i = 1, \dots, d-1$, and 0 otherwise. In particular, E is local (namely isomorphic to $\mathbf{k}[x]/(x^d)$) and $E = \mathbf{k}\ id \oplus \operatorname{rad} E$.*

PROOF. Consider the endomorphism $\zeta = \eta - \lambda\ \mathsf{id}$ and apply the Fitting Lemma. Since ζ is non-invertible, there is n with $\zeta^n = 0$, that is η admits Jordan normal form with unique eigenvalue λ and there is a single block $J = \lambda I + N$ only. Now, E is given by the A such that $AJ = JA$, which is equivalent to the condition $AN = NA$. Since any invariant subspace of N is one of A also, A must be upper triangular and with only a single scalar on each upper diagonal. In other words, from $AN = B = NA$ one has that $a_{i,j-1} = b_{ij} = a_{i+1,j}$ for $i < j$, and 0 as entry otherwise. ☐

If an indecomposable endomorphism (U, η) does not have an eigenvalue in the ground field $\mathbf{k}$, a generalization of Lemma 7.3.3 holds; we recall this in Sections 8.6.2.

We note that, for (U, η) indecomposable, the endomorphism algebra $\operatorname{End}(U, \eta)$ is generated, as a unital $\mathbf{k}$-subalgebra of $\operatorname{End}(U)$, by the single element η (see Proposition 8.6.11). In other words, it consists simply of "polynomials in η". In particular, it is a commutative algebra, and a subspace of U is invariant under $\operatorname{End}(U, \eta)$ if and only if it is invariant under η. Further details about these endomorphism algebras are discussed in Section 8.6.4.

7.4. Decompositions in symplectic spaces

If V carries a symplectic form ω, we define a transpose operation ^{t} on its endomorphisms by the usual formula $\omega(f(x), y) = \omega(x, f^t(y))$. Then the condition $\pi^t\pi = 0$ on an idempotent π means that the image of π is an isotropic subspace. In fact, for any x and y in V, we have $\omega(\pi(x), \pi(y)) = \omega(x, \pi^t\pi(y))$.

Similarly, if $\pi\pi^t = 0$, then the image of π^t is isotropic. It follows that, if $\pi\pi^t = 0 = \pi^t\pi$ and $\pi + \pi^t = 1$, then the images of π and π^t give a decomposition of V as the direct sum of two isotropic subspaces which must be lagrangian and hence in duality by the bilinear form. We state this result in the form of a lemma for use below.

LEMMA 7.4.1. *Decompositions of a symplectic vector space V as a direct sum of (two) lagrangian subspaces are in 1-1 correspondence with idempotent endomorphisms $\pi : V \to V$ such that $\pi^t\pi = 0 = \pi\pi^t$ and $\pi + \pi^t = 1$.*

Similarly, for symplectic direct sum decompositions, we have the following characterization in terms of idempotents.

LEMMA 7.4.2. *Symplectic direct sum decompositions of V into two subspaces are in 1-1 correspondence with idempotent endomorphisms $\pi : V \to V$ which are self-adjoint, i.e. $\pi^t = \pi$.*

PROOF. It is a standard fact that $\operatorname{Ker}(\pi) = (\operatorname{Im} \pi^t)^\perp$, and $\operatorname{Ker} \pi$ is also the image of $1 - \pi$. It follows that $\pi = \pi^t$ if and only if the images of π and $1 - \pi$ are orthogonal, which means that the corresponding direct sum decomposition is symplectic. ☐

Now suppose that φ is a symplectic representation in V of a poset with involution $(P, \perp)$.

LEMMA 7.4.3. *If f is an endomorphism of φ, then f^t is an endomorphism of φ as well.*

PROOF. We must show that, for each $p \in P$, $f^t(\varphi(p)) \subseteq \varphi(p)$. This is equivalent to showing that, for $a \in \varphi(p)$ and $b \in \varphi(p)^\perp$, we have $\omega(f^t(a), b) = 0$, or, equivalently, $\omega(a, f(b)) = 0$. Since φ is symplectic, we have $b \in \varphi(p^\perp)$; since f is an endomorphism of φ, $f(b) \in \varphi(p^\perp) = \varphi(p)^\perp$ as well. □

LEMMA 7.4.4. *Symplectic direct sum decompositions of φ into two representations are in* 1-1 *correspondence with self-adjoint idempotents $\pi = \pi^t$ which are endomorphisms of the underlying linear representation, i.e. $\pi \in End(\hat{\varphi})$.*

PROOF. A direct sum decomposition $V = V_1 \oplus V_2$ of φ is nothing else than a decomposition of $\hat{\varphi}$ where the summands V_1 and V_2 are also mutual orthogonals. Such a decomposition corresponds to an idempotent $\pi \in \mathrm{End}(\hat{\varphi})$ such that $\pi^t = \pi$ (c.f. Lemma 7.4.2). □

LEMMA 7.4.5. *If φ is an indecomposable symplectic representation and if $f \in End(\hat{\varphi})$ is an endomorphism of $\hat{\varphi}$ such that $f^t = \pm f$, then f is either nilpotent or an isomorphism.*

PROOF. From the proof of Lemma 7.2.1 we know that there exists a non-negative integer d such that $V = \mathrm{Ker} f^d \oplus \mathrm{Im} f^d$ is a decomposition of $\hat{\varphi}$. Since $(f^d)^t = (f^t)^d = \pm f^d$, this decomposition of V is one into orthogonal summands; hence it is a symplectic direct sum decomposition of φ. Since φ is assumed to be indecomposable as a symplectic representation, either $\mathrm{Ker} f^d$ or $\mathrm{Im} f^d$ must be zero. □

7.5. Duals, Compatible forms

Let $\psi : P \to \Sigma(V)$ be any linear representation in V of a poset P equipped with an order-reversing involution $\perp$. We can think of ψ as a linear representation of $(P, \perp)$ which doesn't "see" the involution. Define the **dual representation** of ψ in V^* by

$$\psi^* : (P, \perp) \longrightarrow \Sigma(V^*), \quad \psi^*(p) = \psi(p^\perp)^\circ \quad \forall p \in P. \tag{252}$$

Note that this definition makes use of the involution on P, i.e. ψ^* is the dual of ψ *with respect to the involution* $\perp$ on P. In particular, the definition only makes sense viewing ψ as a linear representation of $(P, \perp)$ (rather than only of P). The combination of two order inversions - once due to the poset involution and once due to the annihilator operation - leads to ψ^* being order preserving.

By Lemma 7.1.1, if φ is a symplectic representation of $(P, \perp)$ in (V, ω), then $\tilde{\omega}$ is an isomorphism from $\hat{\varphi}$ to $\hat{\varphi}^*$. This shows that a representation of P can be compatible with a symplectic structure only if it is self-dual, i.e. isomorphic to its dual. In particular, it has a dimension vector which is **self-dual** in the sense that $\dim \varphi(p) + \dim \varphi(p^\perp) = \dim V$ for any $p \in P$.

The results above lead to the natural question of determining the relation between isomorphisms of a linear representation to its dual, the self-duality of its dimension vector, and the existence (and uniqueness) of compatible symplectic structures.

We will see later that many representations of interest to us are characterized up to isomorphism by their dimension vectors. Hence we record the following simple observation.

LEMMA 7.5.1. *If a representation ψ is isomorphic to its dual, then it has a self-dual dimension vector. If a representation ψ is characterized up to isomorphism by its dimension vector, and this dimension vector is self-dual, then ψ is isomorphic to its dual.*

PROOF. The first statement has already been noted. For the second, we observe that, if the dimension vector of ψ is self-dual, then ψ and ψ^* have the same dimension vector, and so by assumption they must be isomorphic. □

In studying symplectic structures, it will sometimes be important to consider non-degenerate symmetric bilinear forms, as well. For these, there is an analogous notion of orthogonality and, therefore, an analogous notion of representation of a partially ordered set with involution. To capture both kinds of forms, we'll speak of ε-**symmetric** bilinear forms, where $\varepsilon = 1$ for symmetric forms and $\varepsilon = -1$ for antisymmetric forms; similarly, an ε-**symmetric representation** of a poset with involution is the generalization of the definition of symplectic representation to ε-**symmetric** forms. We will often identify a non-degenerate bilinear form B on V with the linear isomorphism $B : V \to V^*, v \mapsto B(v, \cdot)$, setting $B^* : V \to V^*, v \mapsto B(\cdot, v)$. With this notation, B is ε-symmetric if and only if $B^* = \varepsilon B$[4]. In referring to the parity ε of B we sometimes write $\varepsilon(B)$.

For a fixed linear representation ψ on V of a poset $(P, \perp)$ with involution, one can ask how many different non-degenerate ε-symmetric forms B exist (if any) which are compatible with ψ in the sense that

$$\psi(p^\perp) = \psi(p)^\perp \quad \forall p \in P,$$

where the involution $\perp$ on V is the one induced by B. A non-degenerate ε-symmetric form which is compatible with ψ in this sense will be called a **compatible form**.

LEMMA 7.5.2. *Let ψ be a linear representation of $(P, \perp)$ in V, and $B : V \to V^*$ a non-degenerate ε-symmetric form. Then B is a compatible form (for ψ) if and only if B is an isomorphism $\psi \to \psi^*$.*

PROOF. That B is compatible means that $\psi(p^\perp) = \psi(p)^\perp$ for all $p \in P$. This is equivalent with $B(\psi(p^\perp)) = \psi(p)^\circ$, and since $\psi(p)^\circ = \psi^*(p^\perp)$, this is the same as $B(\psi(p^\perp)) = \psi^*(p^\perp)$ for all $p \in P$. □

The following is Proposition 2.5 (2) in [**QSS79**].

LEMMA 7.5.3. *Let ψ be an indecomposable linear representation in V of a poset with involution $(P, \perp)$. Then ψ is isomorphic to its dual if and only if there exists a compatible form.*

PROOF. If there exists a compatible form, then by Lemma 7.5.2, such a form defines an isomorphism between ψ and ψ^*.

Conversely, suppose that $B : \psi \to \psi^*$ is an isomorphism. Then so is B^*, and hence the symmetric and antisymmetric parts $(B + B^*)/2$ and $(B - B^*)/2$ of B are also morphisms of representations, as are the endomorphisms $B^{-1}(B + B^*)/2$ and $B^{-1}(B - B^*)/2$, whose sum is the identity morphism. By Lemma 7.2.1, the ring of endomorphisms of ψ is local, so the two summands cannot both be degenerate. It follows that either the symmetric or antisymmetric part of B gives a compatible non-degenerate bilinear form. □

[4]Note that if $B : V \to V^*$ is a non-degenerate bilinear form such that there exist no $\lambda \in \mathbf{k}$ such that $B^* = \lambda B$, then in general, for a given subspace $A \subseteq V$, one no longer has "the" orthogonal "$A^\perp$", but rather one must consider the right- and left-orthogonal of A, which in general will not coincide.

7.6. Symplectification

DEFINITION 7.6.1. *The* ***symplectification***[5] ψ^- *of a linear representation* ψ *is the representation*

$$\psi^- : (P, \perp) \longrightarrow \Sigma(V^* \oplus V, \Omega), \quad \psi^-(x) = \psi^*(x) \oplus \psi(x),$$

where $V^* \oplus V$ *is endowed with the canonical symplectic structure*

$$\Omega((\xi, v), (\eta, w)) := \xi(w) - \eta(v) \qquad \xi, \eta \in V^* \quad v, w \in V.$$

PROPOSITION 7.6.2. ψ^- *is a symplectic representation.*

PROOF. For any $p \in P$, we have $\psi^-(p^\perp) = \psi^*(p^\perp) \oplus \psi(p^\perp) = \psi(p)^\circ \oplus \psi(p^\perp)$. This is the symplectic orthogonal of $\psi(p^\perp)^\circ \oplus \psi(p) = \psi^*(p) \oplus \psi(p) = \psi^-(p)$. □

Here are some fundamental properties of the symplectification operation.

First of all, given a symplectic vector space $V = (V, \omega)$, we denote by $\overline{V}$ the symplectic vector space $(V, \overline{\omega})$, where we define $\overline{\omega} := -\omega$. Given a symplectic poset representation φ on (V, ω), we define a symplectic poset representation $\overline{\varphi}$ on $\overline{V}$ by setting $\overline{\varphi}(x) = \varphi(x)$ for all $p \in P$, i.e. as morphisms of posets, φ and $\overline{\varphi}$ are the same, only the form on V has been changed. Given a symplectic poset representation φ, recall that its underlying linear representation is denoted $\hat{\varphi}$.

PROPOSITION 7.6.3. *For any symplectic representation* φ *on* V*, the symplectic representations* $\hat{\varphi}^- = \hat{\varphi}^* \oplus \hat{\varphi}$ *on* $V^* \oplus V$ *and* $\varphi \oplus \overline{\varphi}$ *on* $V \oplus \overline{V}$ *are isomorphic.*

PROOF. An isomorphism of symplectic representations is given by

$$\tau : \varphi \oplus \overline{\varphi} \to \hat{\varphi}^* \oplus \hat{\varphi}, \ (v, w) \mapsto (\tfrac{1}{2}\tilde{\omega}(v + w), v - w).$$

Indeed, τ is a morphism of representations, since when $(v, w) \in \varphi(x) \oplus \overline{\varphi}(x) = \varphi(x) \oplus \varphi(x)$, then $\tau(v, w)) \in \tilde{\omega}(\varphi(x)) \oplus \varphi(x) = \varphi(x^\perp)^\circ \oplus \varphi(x) = \hat{\varphi}^-(x)$. And τ is a symplectic isomorphism, since

$$\begin{aligned}
\Omega(\tau(v, w), \tau(v', w')) &= \tfrac{1}{2}\tilde{\omega}(v + w)(v' - w') - \tfrac{1}{2}\tilde{\omega}(v' + w')(v - w) \\
&= \tfrac{1}{2}[\omega(v, v') + \omega(v, -w') + \omega(w, v') + \omega(w, -w') \\
&\quad - \omega(v', v) - \omega(v', -w) - \omega(w', v) - \omega(w', -w)] \\
&= \tfrac{1}{2}[2\omega(v, v') - 2\omega(w, w')] \\
&= \omega \oplus \overline{\omega}((v, v'), (w, w')).
\end{aligned}$$

□

REMARK 7.6.4. The symplectic isomorphism used in the proposition above is the same as the one behind the Weyl symbol calculus for pseudodifferential operators (c.f. for instance [**DdGP13**], formula (7) and Theorem 8) and the definition of "Poincare's generating function" in hamiltonian mechanics (c.f. [**Wei72**]).

[5]This construction is sometimes known in a more general setting as **hyperbolization** (see Section 4.2, and also [**QSS79**]) because the analogue for symmetric bilinear forms leads to isotropic subspaces in spaces with forms of signature zero, sometimes called "hyperbolic". We use "symplectification" rather than "symplectization" because the latter term already refers to the construction of symplectic manifolds from contact manifolds by adding one dimension.

PROPOSITION 7.6.5. *The symplectification ψ^- of an indecomposable linear representation ψ is symplectically decomposable if and only if ψ admits a compatible symplectic structure.*

PROOF. $\varphi = \psi^-$ is by definition decomposed linearly into the indecomposables ψ^* and ψ. Suppose that it is also symplectically decomposable into two symplectic representations, φ_1 and φ_2. The latter decomposition is also a linear decomposition of $\hat{\varphi}$, and so by Theorem 7.2.3 the linear representations φ_1 and φ_2 must be isomorphic to ψ^* and ψ in some order. In particular ψ (as does ψ^*) admits a compatible symplectic structure.

Conversely, suppose that φ admits a compatible symplectic structure. Then, by Proposition 7.6.3, $(\hat{\varphi})^+ = (\hat{\varphi})^* \oplus \hat{\varphi}$ and $\varphi \oplus \overline{\varphi}$ are isomorphic symplectic representations. Since the latter is decomposable, so is the former. □

PROPOSITION 7.6.6. *If ψ_1 and ψ_2 are indecomposable linear representations, then ψ_1 is isomorphic to ψ_2 or to ψ_2^* if and only if the symplectifications ψ_1^- and ψ_2^- are isomorphic as symplectic representations.*

In particular, two symplectifications of indecomposable linear representations are isomorphic as symplectic representations if and only if they are isomorphic as linear representations.

PROOF. If ψ_1 is isomorphic to ψ_2 or to ψ_2^*, then ψ_1^- is isomorphic to ψ_2^- or to $(\psi_2^*)^-$. If the former holds, we are done. For the latter, we must show that ψ_2^- is isomorphic to $(\psi_2^*)^-$. They are clearly isomorphic as linear representations, but under the isomorphism which exchanges the summands, the symplectic structures differ by a factor of -1. To correct for this factor, we compose with the antisymplectic isomorphism $(\xi, v) \mapsto (-\xi, v)$ from ψ_2^- to itself.

Conversely, if $\psi_1^*(x) \oplus \psi_1(x)$ and $\psi_2^*(x) \oplus \psi_2(x)$ are isomorphic as symplectic representations, then they are in particular also isomorphic as linear representations. This implies that their indecomposable summands are isomorphic in some order, so either $\psi_1 \simeq \psi_2$ or $\psi_1 \simeq \psi_2^*$ □

EXAMPLE 7.6.7. The following are the symplectifications of the indecomposable representations of the poset **2**, i.e. nested pairs of subspaces. Each pair is contained in $\mathbf{k}$, so the symplectification is contained in $\mathbf{k}^* \oplus \mathbf{k}$ and is symplectically indecomposable.

- The symplectification of $\mathbf{k} \supseteq 0$ is $\mathbf{k}^* \oplus \mathbf{k} \supseteq 0 \oplus 0$.
- The symplectification of $0 \supseteq 0$ is $\mathbf{k}^* \oplus 0 \supseteq \mathbf{k}^* \oplus 0$.
- The symplectification of $\mathbf{k} \supseteq \mathbf{k}$ is $0 \oplus \mathbf{k} \supseteq 0 \oplus \mathbf{k}$.

The first example is self-dual, while the latter two examples are dual to one another, and their symplectifications are isomorphic by a "90-degree rotation".

Given a pair of linear poset representations, we say that they are **mutually dual** or, synonymously, a **dual pair** if each representation is isomorphic to the dual of the other. On the level of isomorphism classes, symplectification builds symplectic representations by taking the direct sum of dual pairs of linear representations.

7.7. Relating symplectic and linear indecomposability

The following lemma, due to Quebbemann et al. [**QSS79**] (Thm.3.3) and Sergeichuk [**Ser87**] (Lemma 2) will be an essential tool. It shows that symplectically indecomposable but linearly decomposable representations arise only through symplectification. The analogous result holds in the symmetric setting.

LEMMA 7.7.1. *Suppose that $\varphi : (P, \perp) \to \Sigma(V)$ is an indecomposable symplectic representation such that $\hat{\varphi}$ is (linearly) decomposable. Then there exists an indecomposable linear representation ψ such that $\varphi \simeq \psi^-$.*

PROOF. Because $\hat{\varphi}$ is linearly decomposable, there exists a non-trivial idempotent $\pi_1 \in \mathrm{End}(\hat{\varphi})$. After two modifications, π_1 will be conjugated into an idempotent endomorphism π satisfying the hypotheses $\pi^t\pi = 0 = \pi\pi^t$ and $\pi + \pi^t = 1$ of Lemma 7.4.1, giving the required decomposition.

By Lemma 7.4.3, the idempotent π_1^t is also an endomorphism. By Lemma 7.4.4, $\pi_1^t \neq \pi_1$, since otherwise φ would be decomposable as a symplectic representation. Set $\rho_1 = \pi_1\pi_1^t$. Note that ρ_1 is self-adjoint and lies in $\mathrm{End}(\hat{\varphi})$. By Lemma 7.4.5, ρ_1 must be either nilpotent or an isomorphism. But ρ_1 cannot be an isomorphism, since π_1 and π_1^t have nontrivial kernels and cokernels. So ρ_1 must be nilpotent.

Now set $h_1 := s(\rho_1)$, where $s(X)$ is the binomial series for $(1 - X)^{1/2}$; $s(\rho_1)$ is well-defined because ρ_1 is nilpotent, which implies that the power series is just a polynomial in ρ_1. Note that $h_1 \in \mathrm{End}(\hat{\varphi})$, and that h_1 is also self-adjoint. Furthermore, h_1 is invertible, its inverse being defined by substituting ρ_1 in the binomial series for $(1 - X)^{-1/2}$.

Define $\pi_2 := h_1\pi_1h_1^{-1}$, and note that π_2 lies in $\mathrm{End}(\hat{\varphi})$ and is again a non-trivial idempotent. Furthermore,

$$\pi_2^t\pi_2 = h^{-1}\pi_1h_1^2\pi_1^th_1^{-1} = h_1^{-1}\pi_1(1 - \pi_1\pi_1^t)\pi_1^th_1^{-1} = h_1^{-1}(\pi_1\pi_1^t - \pi_1\pi_1^t)h_1^{-1} = 0.$$

We are half-way there. Now $\rho_2 := \pi_2\pi_2^t$ is a nilpotent, self-adjoint element of $\mathrm{End}(\hat{\varphi})$, and $h_2 := s(\rho_2)$ is again an invertible, self-adjoint endomorphism of $\hat{\varphi}$. Then $\pi := h_2^{-1}\pi_2h_2 \in \mathrm{End}(\hat{\varphi})$ is a non-trivial idempotent such that

$$\pi\pi^t = h_2^{-1}\pi_2h_2^2\pi_2^th_2^{-1} = h_2^{-1}\pi_2(1 - \pi_2\pi_2^t)\pi_2^t\tilde{h}^{-1} = h_2^{-1}(\pi_2\pi_2^t - \pi_2\pi_2^t)h_2^{-1} = 0$$

and

$$\begin{aligned}\pi^t\pi &= h_2\pi_2^t(h_2^{-2})\pi_2h_2 = h_2\pi_2^t(1 - \pi_2\pi_2^t)^{-1}\pi_2h_2 \\ &= h_2\pi_2^t(1 + \pi_2\pi_2^t)\pi_2h_2 = h_2(\pi_2^t\pi_2 + \pi_2^t\pi_2\pi_2^t\pi_2)h_2 = 0,\end{aligned}$$

since $\pi_2^t\pi_2 = 0$. Furthermore, $\pi + \pi^t \in \mathrm{End}(\hat{\varphi})$ is idempotent: $(\pi + \pi^t)^2 = \pi^2 + \pi^t\pi + \pi\pi^t + (\pi^t)^2 = \pi + \pi^t$. But $\pi + \pi^t$ is also self-adjoint, so, by Lemma 7.4.4, $\pi + \pi^t$ must be a trivial idempotent. It cannot be that $\pi + \pi^t = 0$, since this would imply $\pi^t = -\pi$, whence $0 = \pi^t\pi = -\pi^2 = \pi$, a contradiction to $\pi \neq 0$. Thus $\pi + \pi^t = 1$. □

COROLLARY 7.7.2. *If two symplectically indecomposable but linearly decomposable representations are isomorphic as linear representations, then they are isomorphic as symplectic representations.*

PROOF. By Lemma 7.7.1, each of the two symplectic representations is the symplectification of an irreducible linear representation. Since the two representations are linearly isomorphic, by Lemma 7.6.6, they are symplectically isomorphic. □

Lemma 7.7.1 tells us that every indecomposable symplectic representation is either linearly indecomposable or the symplectification of a linearly indecomposable representation, but not both. In the former case we say that φ is of **non-split** type; in the latter case we say that φ is of **split** type.

7.8. Uniqueness of compatible forms

We briefly discuss the question of uniqueness of compatible bilinear forms and formulate a lemma which will be the basis of our analysis of uniqueness of compatible forms. Later we will verify the hypothesis of the lemma for the representations which concern us, and one particular type will require a generalization. The proof uses ideas from that of Lemma 5 in [**Ser87**]; it is interesting that the proof uses a version of the "square root" construction of Lemma 2 of that paper (which is our Lemma 7.7.1). In addition, we use an idea (simplified for our context) from Proposition 2.5 in [**QSS79**] when showing that any two compatible forms must both be symmetric or antisymmetric.

LEMMA 7.8.1. *Let ψ be a linearly indecomposable representation in V of an involutive poset for which the endomorphism algebra E (which is local by Corollary 7.2.1) and has the property that $E = \mathbf{k}id \oplus RadE$. If ψ admits two compatible bilinear forms, then these forms are equal up to a constant scalar multiple and an automorphism of ψ. In particular, the forms must both be symmetric or antisymmetric.*

PROOF. If B_1 and B_2 are the isomorphisms from ψ to ψ^* corresponding to two compatible forms, then $C = B_1^{-1}B_2$ is an automorphism of ψ.

Let † denote the antiautomorphism of E given by the operation of adjoint with respect to B_1, i.e. $B_1(A^\dagger x)(y) = B_1(x)(Ay)$, which is equal to $A^*(B_1(x))(y)$, so we have $A^\dagger = B_1^{-1}A^*B_1$.

Define the signs ε_1 and ε_2 by $B_i^* = \varepsilon_i B_i$, and let $\epsilon = \varepsilon_1\varepsilon_2$. Then we have $C^\dagger = \epsilon C$. In fact,

$$C^\dagger = (B_1^{-1}B_2)^\dagger = B_1^{-1}B_2^*(B_1^{-1})^*B_1 = \epsilon B_1^{-1}B_2B_1^{-1}B_1 = \epsilon B_1^{-1}B_2 = \epsilon C.$$

By our hypothesis, we may write C as $c\mathsf{id} - r_1$, where $r_1 \in R$ and c is a scalar and $c \neq 0$ since C is invertible. By replacing B_2 by $c^{-1}B_2$ and repeating the argument up to this point, we may assume that $c = 1$, so that $C = 1 - r$ for an $r \in R$.

Now if ϵ were equal to -1, we would have

$$1 - r^\dagger = (1-r)^\dagger = -(1-r),$$

which would imply that $1+1 = r^\dagger - r$, which is impossible since R is closed under addition and contains no invertible elements (and, by assumption, $2 \neq 0$ in our ground field). So $\epsilon = 1$, and both forms are either symmetric or antisymmetric. Furthermore, $r = r^\dagger$.

By Lemma 7.2.1, we know that r is nilpotent, and so we can use the formal power series for $\sqrt{1-r}$ to construct an automorphism h such that $h^\dagger = h$ and $h^2 = C$.

Now we have

$$h^*B_1h = B_1h^\dagger B_1^{-1}B_1h = B_1h^\dagger h = B_1h^2 = B_1C = B_2,$$

which shows that h is an isomorphism between the bilinear forms B_1 and B_2. □

REMARK 7.8.2. A form ω on V and its scalar multiple $a\omega$ are equivalent by a homothety of V (which is automatically an automorphism of any poset representation) if and only if a is a square in the coefficient field. This means that the set of equivalence classes of compatible forms under homothety is a principal homogeneous space of the **square class group** of $\mathbf{k}$, defined as the quotient $\mathbf{k}^\times/{\mathbf{k}^\times}^2$ of the multiplicative group of nonzero elements of $\mathbf{k}$ by the perfect squares.

Even if a is not a square, ω and $a\omega$ might, a priori, still be isomorphic by a linear isomorphism which preserves a particular poset representation. This may also be the case if the representation is linearly decomposable, though in the present work we will not consider the question of compatible forms for decomposable linear representations.

REMARK 7.8.3. In Section 8.3.5 we will see that certain self-dual poset representations fulfill the hypotheses of Lemma 7.8.1. In these cases it turns out that compatible forms which differ by a scalar $c \in \mathbf{k}$ are in fact equivalent if and only if c is a square.

For certain other self-dual poset representations, however, only a generalization of Lemma 7.8.1 applies and both symmetric and antisymmetric compatible forms exist for a given such representation; see in particular Sections 8.6.5 and 8.6.6.

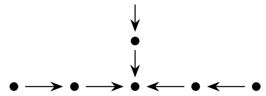
CHAPTER 8

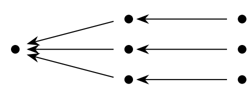
Isotropic triples

We transition now from the general theory of symplectic poset representations to an analysis of the indecomposable symplectic poset representations connected to a specific classification problem. In [**LW15**], the authors give a complete classification of pairs of coisotropic subspaces in Poisson vector spaces and, equivalently by duality, pairs of isotropic subspaces in presymplectic vector spaces. Each such pair is uniquely (up to isomorphism and ordering) decomposable as a direct sum of multiples of ten indecomposable pairs, for which there are simple normal forms in ambient spaces of dimension 1, 2, or 3. These decomposition problems are special cases of the problem of classifying triples of coisotropic or isotropic subspaces in symplectic vector spaces, with an extra condition relating the third subspace to the first two.

In this chapter, we will deal with *all* (co)isotropic triples in symplectic vector spaces. The decomposition into indecomposables is still possible with summands essentially unique, but there are many more indecomposables. In dimensions up to 4, there are still only finitely many isomorphism classes of indecomposables, while in higher dimensions the moduli space of such classes includes parametrized families as well as single points which may or may not be in the closure of such families.

The classification of pairs in [**LW15**] was done by elementary arguments in linear algebra, but the results there (as well as those in [**LW16**] on the classification of (co)isotropic relations) suggested links with the representation theory of quivers, particularly of those associated with (extended) Dynkin diagrams and the closely related representation theory of partially ordered sets (posets). (See e.g. Gabriel and Roiter [**KS92**].) We rely on these to carry out the classification of triples. In fact, we largely reduce our problem to that of studying representations which are maps from a certain 6-element poset with involution to the poset of subspaces of a symplectic vector space with the involution given by symplectic orthogonality. The classification of these representations, without the involution, is essentially that of certain representations of a quiver associated to the extended Dynkin diagram $\tilde{E}_6$, which is a tree consisting of a central vertex attached to three "branches" containing two vertices each. The quiver in question is obtained from $\tilde{E}_6$ by orienting all of its edges toward the central vertex. Here are two depictions of this quiver; the first is the most common, while we will use the latter, to emphasize the poset structure:





For our results, we rely on the classification of representations of extended Dynkin quivers given in [**DR76**] and [**DF73**], for the case of $\tilde{E}_6$. Representations of this particular quiver have also been studied in quite some detail by Stekolchshik, see e.g. [**Ste04**] and [**Ste07**]. The study of poset representations in spaces equipped with an (anti-)symmetric inner product was first developed, to our knowledge, by Scharlau and collaborators; see [**Sch75**] for a concise and enlightening overview.

For (co)isotropic triples, the connection with the $\tilde{E}_6$-type quiver described above is this: the central vertex corresponds to the ambient symplectic vector space, while each branch corresponds to an isotropic subspace and (adjacent to the central vertex) the coisotropic orthogonal in which it is contained.

The associated six-element poset consists of the vertices in the branches, with partial ordering given by the arrows connecting them, and (order-reversing) involution given by exchanging the elements of each pair. We will use one of the standard notations, $\mathbf{2}+\mathbf{2}+\mathbf{2}$, for this poset.

Since the operation of "taking the symplectic orthogonal" induces a one-to-one correspondence between isotropic subspaces and coisotropic ones, in the following we focus on and refer simply to isotropic triples. The concomitant results for coisotropic triples are implicit.

A crucial part of our analysis is a result due to Quebbemann, Scharlau, and Schulte [**QSS79**] and Sergeichuk [**Ser87**], which is formulated for poset representations in Lemma 7.7.1. For isotropic triples, this Lemma says that every indecomposable isotropic triple in dimension $2n$ is either already indecomposable as a linear representation of the poset $\mathbf{2}+\mathbf{2}+\mathbf{2}$ or is obtained from an indecomposable linear representation of the same poset in dimension n by a "doubling construction" known as hyperbolization ([**QSS79**, p. 267]), and which in our context we will call **symplectification**. (It is closely connected to the cotangent bundle construction in symplectic geometry, though the latter always produces lagrangian subspaces.) This dichotomy reduces our problem to deciding which such indecomposable linear representations actually come from indecomposable isotropic triples, and finding the nonisomorphic isotropic triples which may give rise to the same indecomposable linear representation.[1]

8.1. Resumé

In this section we give a summary of our results. We hope that placing this summary here, rather than as a final section, will give the reader an initial rough overview and a place to refer back to. We include a short list of our most essential terms and notions:

- A **linear poset representation** (or just "poset representation") of a poset P in a vector space V (always assumed finite-dimensional) is an order-preserving map from P to the poset of subspaces of V. Poset representations will usually be denoted by the letter ψ. Representations of the poset $\mathbf{2}+\mathbf{2}+\mathbf{2}$ are called sextuples.
- If a poset P is equipped with an order-reversing involution, then to each representation ψ of P in V, there is a **dual** representation ψ^* of P in V^*. (For the definition, see (252) in Section 7.5.) In the present work, we always assume the poset $\mathbf{2}+\mathbf{2}+\mathbf{2}$ to be equipped with the order-reversing involution that exchanges the two elements of each of the three ordered pairs.

[1]The simplest triple illustrating this possibility consists of three distinct lines in a symplectic plane. These are lagrangian, so each one corresponds to both the isotropic and coisotropic subspaces in a nested pair. If we forget the symplectic structure, there is no further invariant, but in the case of real coefficients, there is a symplectic invariant given by the cyclic order of the three lines with respect to the symplectic orientation. This is sometimes called the Maslov index or Kashiwara-Vergne index of the lagrangian triple.

- An **isotropic triple** φ is an ordered triple of isotropic subspaces of a symplectic vector space. The three isotropics, together with their corresponding symplectic orthogonals, form a sextuple of subspaces which define a linear representation of the poset $\mathbf{2}+\mathbf{2}+\mathbf{2}$, the underlying linear representation $\hat{\varphi}$ of the triple. The underlying poset representation $\hat{\varphi}$ of an isotropic triple is necessarily isomorphic to its dual; i.e $\hat{\varphi}$ is "self-dual".
- Given a sextuple ψ in V, a **compatible symplectic form** is a symplectic form on V with respect to which ψ becomes the underlying poset representation of an isotropic triple. (See Section 7.5.)
- Isotropic triples are examples of **symplectic poset representations** (see Definition 7.1.2). There are distinct notions of decomposition (and of "indecomposable") for linear poset representations and for symplectic poset representations, respectively (the latter are *orthogonal* decompositions.) In particular, an indecomposable isotropic triple may be decomposable as a linear poset representation.
- Any representation ψ of a poset P has an associated **dimension vector**: this is the function which assigns to each element $x \in P$ the dimension of the associated subspace $\psi(x)$. We usually write the dimension vector not as a function, but as a tuple (i.e. a "vector").
- Throughout we assume that the ground field $\mathbf{k}$ is perfect and does not have characteristic 2. The condition of "perfectness" is not a strong one: perfect fields include fields which are algebraically closed, fields of characteristic zero, and finite fields.

Resumé:

(1) Symplectically indecomposable isotropic triples φ come in two kinds:
- Split: the underlying sextuple $\hat{\varphi}$ of the isotropic triple is decomposable as a linear poset representation. By Lemma 7.7.1, φ is isomorphic to the symplectification of some indecomposable poset representation ψ of $\mathbf{2}+\mathbf{2}+\mathbf{2}$ which is not self-dual; in particular $\hat{\varphi} \cong \psi \oplus \psi^*$.
- Non-split: φ is such that $\hat{\varphi}$ indecomposable and self-dual

Our main work is the identification of the duals of indecomposable sextuples and the classification of the indecomposable non-split isotropic triples. Once this is done, by Lemma 7.7.1, the classification of the split-type indecomposable isotropic triples is essentially automatic. Throughout, it is understood that when we speak of classification of indecomposables, we mean the classification of *isomorphism classes* of indecomposables.

(2) In order to classify the non-split indecomposable isotropic triples, we first identify which sextuples ψ are self-dual. We have the following subcases (we use labels (a) through (e), which are also referred to further below):
- Discrete-type: in these cases, ψ is based on an indecomposable nilpotent endomorphism. Up to isomorphism, ψ is uniquely determined by its dimension vector $\mathbf{d}$. There are the following types, with $k \in \mathbb{Z}_{\geq 0}$:
  (a) $A(3k+1, 0)$ with $\mathbf{d} = (3k+1; 2k+1, k, 2k+1, k, 2k+1, k)$,
  (b) $A(3k+2, 0)$ with $\mathbf{d} = (3k+2; 2k+1, k+1, 2k+1, k+1, 2k+1, k+1)$.
- Continuous-type: in these cases, ψ has a dimension vector of the form $(3k, 2k, k, 2k, k, 2k, k)$, with $k \in \mathbb{Z}_{>0}$, and is based on an indecomposable endomorphism η (this is a generalization of the $\Delta(k, \lambda)$ of Donovan-Freislich

[**DF73**], and we also use normal forms based on the homogeneous representations of Dlab-Ringel [**DR76**]). In our encoding, the underlying endomorphism η of a self-dual ψ is necessarily such that η is similar to $(\mathsf{id} - \eta)^*$. The following types of endomorphism η (up to similarity) account for all the indecomposable self-dual continuous-type sextuples:
If η has an eigenvalue λ in the base field:
  (c) $\lambda = \frac{1}{2}$.
If η has has no eigenvalue in the base field:
  (d) Over the reals: the complex eigenvalues of η are $\frac{1}{2} \pm b_\eta\sqrt{-1}$ for unique real $b_\eta > 0$.
  (e) Over perfect fields in general: $\eta = \frac{1}{2}\mathsf{id} + \zeta$ where $\zeta \neq 0$ is similar to $-\zeta^*$. The characteristic (= minimal) polynomial of ζ is of the form $r(x)^m$ for an irreducible polynomial r which is of the form $r(x) = p(x^2)$ for some polynomial p.

(3) With the indecomposable self-dual sextuples classified, we then determine which of these admit compatible symplectic forms (and we give such forms explicitly via coordinate matrices). We find that compatible symplectic forms exists as follows:
  (a) for $A(3k + 1, 0)$ if and only if k is odd.
  (b) for $A(3k + 2, 0)$ if and only if k is even.
  (c) for η having an eigenvalue in $\mathbf{k}$: if and only if k is even.
  (d-e) for η having no eigenvalue in $\mathbf{k}$: for all k.

(4) Following the question of existence of compatible symplectic forms for indecomposable self-dual sextuples, we then address the question of uniqueness. We find that compatible symplectic forms for a sextuple ψ are unique up to isomorphism (i.e. up to an isometry which is an automorphism of ψ) and, in
  (a-c), multiplication by a scalar. There are no compatible symmetric forms.
  (d), multiplication by a scalar. There are also compatible symmetric forms.
  (e), multiplication by a 'scalar' $Z \in F_H^+$. Here $F = \mathbf{k}[x]/q(x)$ is considered as a subring of $\mathbf{k}^{3k\times 3k}$ and F_H^+ consists of all $Z \in F$ such that $Z^t H = HZ$, where H is the coordinate matrix of a particular compatible form. There are also compatible symmetric forms.

(5) The following provides a complete list of isomorphism types of isotropic triples in the non-split case. Given a non-split isotropic triple φ with symplectic form ω (and associated coordinate matrix H) there is an automorphism of $\hat{\varphi}$ which is
  (a-d) an isometry from ω to $c\omega$, for some $0 \neq c \in \mathbf{k}$, if and only if c is a square in $\mathbf{k}$. Thus, compatible symplectic forms for a given sextuple $\psi = \hat{\varphi}$ are parametrized by the square class group $\mathbf{k}^\times/(\mathbf{k}^\times)^2$.
  (e) an isometry from H to HZ, for some $0 \neq Z \in F$, if and only if $Z = X^2 - Y^2$ where $X \in F_H^+$ and $Y \in F_H^-$. Here, $F_H^\pm = \{Z \mid Z^t H = \pm HZ\}$.

(6) For a fixed indecomposable sextuple ψ, the set (if non-empty) of all compatible symplectic forms[2] is given via linear expressions involving
  (a) k parameters,
  (b) $k + 1$ parameters,
  (c-d) $\frac{k}{2} + 1$ parameters.

Proof of the Resumé.

[2]Here we mean *all* compatible forms, i.e. not only up to isometry.

(1) By Lemma 7.7.1.
(2) For the cases (a-b), in Proposition 8.2.4 it is stated which discrete-type indecomposable sextuples are self-dual, and in Theorem 8.3.3 it is shown which ones admit compatible symplectic forms.

For continuous-type indecomposable sextuples, it is established in Section 8.4.5 that self-duality only occurs for framed sextuples, and from Proposition 8.4.17 it follows that a framed sextuple $\mathcal{S}_\eta$ is self-dual if and only if η is similar to $\mathrm{id} - \eta^*$. Furthermore, those $\mathcal{S}_\eta$ which underly non-split isotropic triples are identified:
   (c) in Theorem 8.5.1 for the case when the underlying endomorphism has an eigenvalue in the ground field.
   (d) in Theorem 8.5.4 for the case when $\mathbf{k} = \mathbb{R}$ and the underlying endomorphism has no eigenvalue in $\mathbb{R}$.
   (e) in Theorem 8.6.9 for the general case when $\mathbf{k}$ is perfect and the underlying endomorphism has no eigenvalue in $\mathbf{k}$. See also Corrollary 8.6.10.
(3) For (a-b), see Theorem 8.3.3; for (c), see Theorem 8.5.1; (d-e), see Theorem 8.6.9.
(4) For (a-b), see Theorem 8.3.12; for (c), see Theorem 8.5.1; for (d), see Theorem 8.5.4; for (e), see Theorem 8.6.17. The uniqueness statements in the theorems above for (a-d) make use of Lemma 7.8.1, and in the case (e), Theorem 8.6.17 makes use of Lemma 8.6.15.
(5) As in the previous point: for (a-b), see Theorem 8.3.12; for (c), see Theorem 8.5.1; for (d), see Theorem 8.5.4; for (e), see Theorem 8.6.17.
(6) For (a-b), see Remark 8.3.9; for (c-d), see Theorem 8.5.1 and Theorem 8.5.4.

□

Corollary 8.1.1. *For an indecomposable representation ψ of the poset $P = \mathbf{2}+\mathbf{2}+\mathbf{2}$ in a vector space V the following are equivalent*
(1) *There is a symplectic representation φ such that $\hat{\varphi} = \psi$.*
(2) *ψ is self-dual and $\dim V$ is even.*

Remark 8.1.2. The following topics are deferred to possible future work.
- For those representations which do *not* admit compatible symplectic structures (since they are not self-dual or admit only symmetric structure), a detailed description of the isotropic triples resulting from their symplectifications.
- Analysis of how the classification of isotropic triples changes when the ground field is changed.
- A description of the isometry groups of indecomposable isotropic triples.
- A discussion of the classification of isotropic triples within the category-theoretic framework of Quebbemann, Scharlau, Schulte [**QSS79**] and Sergeichuk [**Ser87**].
- A further analysis of the relation between poset representations of $\mathbf{2}+\mathbf{2}+\mathbf{2}$ (sextuples of subspaces) and poset representations of $\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}$ (quadruples of subspaces).
- The study of how a general (not necessarily indecomposable) isotropic triple decomposes into indecomposable summands; in particular the question of how unique such a decomposition is.
- The question of defining invariants for isometry types of isotropic triples which give multiplicities of indecomposable summands (in particular in relation to the perfect elements established by Stekolchshik [**Ste07**]).

- An explanation in further detail of the relation between isotropic triples and linear hamiltonian vector fields.

8.2. Interlude: overview, preview, examples

8.2.1. Next steps. By Theorem 7.2.3 and Lemma 7.7.2 one obtains the following.

COROLLARY 8.2.1. *For a given symplectic representation, consider orthogonal decompositions into symplectically indecomposable summands. Then any summand is either split or non-split and the following are uniquely determined:*

(1) *The isomorphism types and multiplicities of split summands.*
(2) *The linear isomorphism types and multiplicities of non-split summands.*

At this point, we have finished our discussion of the general theory of symplectic poset representations and are ready to move on to specific cases related to isotropic triples. We will do the following things.

- In section 8.2.2 we review some essentials of the theory of quiver representations – as background for the classification results to be used. We explain how the classification, obtained in [**DF73, DR76**], of indecomposable representations of a quiver related to the Dynkin diagram $\tilde{E}_6$ gives the classification of indecomposable linear representations of the poset $\mathbf{2}+\mathbf{2}+\mathbf{2}$.
- As a warmup, we derive, in sections 8.2.3 and 8.2.4, the classification of isotropic k-tuples, for $k = 0, 1, 2$, from the classification of representations of quivers associated with the Dynkin-diagrams A_1, A_3, and A_5. Then in section 8.2.5 we give an overview of the quiver representation classification results for the Dynkin diagram $\tilde{E}_6$, which we will use for classifying isotropic triples. In section 8.2.6, again as a warmup, we discuss isotropic triples in ambient dimension 2, and give, in section 8.2.7, a preview of the situation in higher dimensions.
- Using the results of [**DR76, DF73**], we identify which indecomposable linear representations of $P = \mathbf{2}+\mathbf{2}+\mathbf{2}$ are dual to one another when we equip $\mathbf{2}+\mathbf{2}+\mathbf{2}$ with the involution $\perp$ which exchanges the respective elements of the three nested pairs in $\mathbf{2}+\mathbf{2}+\mathbf{2}$. From now on we will use the term **sextuples** to refer to linear representations of this poset with involution.
- From the general theory we know that self-dual sextuples admit compatible symmetric or symplectic forms (or both). We determine which self-dual sextuples admit compatible symplectic forms (thus giving non-split isotropic triples), and we give explicit constructions of such forms.
- For the self-dual sextuples, we reduce the classification of compatible forms to a field-theoretic description. When $\mathbf{k} = \mathbb{R}$ or when $\mathbf{k}$ is algebraically closed, compatible forms are parametrized by the square class group $\mathbf{k}^\times/(\mathbf{k}^\times)^2$. For general perfect fields, a similar, though slightly more complicated, description is obtained; see Theorem 8.6.17.

8.2.2. Quiver and poset representations. We recall here some basic definitions and results in quiver representation theory, referring to the literature[3] for more details. A **quiver** $\mathcal{Q}$ is simply a directed graph, i.e. a set $\mathcal{V}$ of vertices and a set $\mathcal{A}$ of arrows, with source and target maps s and t from $\mathcal{A}$ to $\mathcal{V}$. We allow multiple arrows with a given source

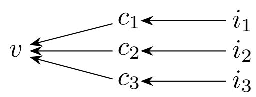
[3]For example {[**Bar15**], [**Ben**], [**DW17**], [**EGH**$^+$**11**], [**Kac80**], [**Sch14**]} is a small sample subset of the available references, to give the reader a starting point.

and target, but assume the sets $\mathcal{A}$ and $\mathcal{V}$ to be finite. With a chosen ground field $\mathbf{k}$, a **representation** ρ of $\mathcal{Q}$ is simply an assignment to each vertex v a (finite dimensional, for our purposes) $\mathbf{k}$-vector space $\rho(v)$ and to each arrow a a linear map $\rho(a) : \rho(s(a)) \to \rho(t(a))$. A morphism μ from ρ_1 to ρ_2 is a family of linear maps $\mu_v : \rho_1(v) \to \rho_2(v)$ making the obvious diagrams commute. When the family of linear maps consists of isomorphisms, then μ is called an isomorphism. The collection of representations of a fixed quiver with their morphisms form a category.

A fundamental problem in the theory of quiver representations (as is the case for representations of just about anything) is to describe the structure of the set of isomorphism classes of representations, and, among these, the indecomposable representations, which are those not decomposable into nontrivial direct sums.

The first basic result is the Krull-Schmidt theorem, which states that each representation of a quiver is isomorphic to a direct sum of indecomposables, and that the summands in this decomposition, with their multiplicities, are unique up to isomorphism and reordering. This reduces the classification of representations to the enumeration of those which are indecomposable. When the set of isomorphism classes of indecomposables is finite, the quiver is of **finite type**.

As mentioned above, we will be studying isotropic triples by considering them as linear representations of the poset $\mathbf{2}+\mathbf{2}+\mathbf{2}$. These poset representations can be identified with particular representations of the following quiver; it is obtained by choosing the following orientation on the extended Dynkin diagram $\tilde{E}_6$:



(253)

We'll refer to this quiver also as $\tilde{E}_6$. The labels on the vertices are in principle arbitrary (and sometimes unnecessary); we fix this choice since it is suggestive for our application to (co)isotropic triples. We identify the vertices other than v with the elements of the poset $\mathbf{2}+\mathbf{2}+\mathbf{2}$, with the partial order indicated by the arrows, i.e. $i_1 \le c_1$, $i_2 \le c_2$ and $i_3 \le c_3$.

When considering representations of this quiver, we will denote the space associated to the vertex v by V, and the spaces associated to the vertices c_1, c_2, c_3 and i_1, i_2, i_3 will be denoted by C_1, C_2, C_3 and I_1, I_2, I_3, respectively. When it is clear what the maps are, a representation of (253) will be denoted by the 7-tuple of spaces $(V; C_1, I_1; C_2, I_2; C_3, I_3)$ or just $(V; C_i, I_i)$. We will also call such representations **sextuples**, just as we do poset representations of $\mathbf{2}$+$\mathbf{2}$+$\mathbf{2}$. Abusing notation slightly, dimension vectors of sextuples will be denoted $(v, c_1, c_2, c_3, i_1, i_2, i_3)$, where the entries denote the dimensions of the (sub)spaces of a representation $(V; C_1, I_1; C_2, I_2; C_3, I_3)$.

It is straightforward to see that, on the level of isomorphism classes, linear poset representations of $\mathbf{2}+\mathbf{2}+\mathbf{2}$ are in one-to-one correspondence with quiver representations of the quiver $\tilde{E}_6$ where all arrows are represented by injective linear maps (we will call these injective representations). This correspondence is compatible with the notions of direct sum for poset and quiver representations, respectively. Furthermore, one can prove that an indecomposable quiver representation of $\tilde{E}_6$ is an injective representation if and only if the space assigned to the central vertex is non-zero (see [**Ste04**], Proposition A.7.1).

Using this, one can read off the indecomposable representations of $\mathbf{2}$+$\mathbf{2}$+$\mathbf{2}$ from those of $\tilde{E}_6$ as given in [**DR76, DF73**]. We will mainly use the explicit normal forms given in [**DR76**]. Since that reference (in contrast to [**DF73**]) only treats the case when the ground field is algebraically closed, we will use a straightforward generalization of their normal forms for more general fields. By inspection of the proofs [**DR76**], it is only the normal forms for continuous-type representations that must be generalized. These are discussed in Section 8.4.

An essential tool for the study of quiver representations is the **Tits form** of a quiver $\mathcal{Q}$. This is the quadratic form q on the $\mathbb{Z}$-module generated by the vertices defined by $q(\mathbf{n}) = \sum_{\mathcal{V}} n_v^2 - \sum_{\mathcal{V}\times\mathcal{V}} a_{v,w} n_v n_w$, where n_v is the coefficient of v in $\mathbf{n}$, and $a_{v,w}$ is the number of arrows from v to w. Note that this form does not depend on the direction of the arrows.

The idea behind the Tits form is that, if the coefficients of $\mathbf{n}$ are the dimensions of vector spaces assigned to the vertices, then the second term is (the negative of) the dimension of the linear space of all representations of $\mathcal{Q}$ in this family of vector spaces, while the first term is the dimension of the group, acting on the space of representations, whose orbits are the isomorphism classes. In fact, the scalar multiples of the identity act trivially, so we may say that the "virtual dimension" of the moduli space of isomorphism classes of representations with dimension vector $\mathbf{n}$ is $1 - q(\mathbf{n})$. The actual dimension is at least this large (and larger if more of the group acts trivially), so if our ground field is, e.g., the real or complex numbers, the only way in which there can be finitely many isomorphism classes with dimension vector $\mathbf{n}$ is if $q(\mathbf{n})$ is at least 1.

This suggests (but does not prove, for various reasons), that a quiver is of finite type if and only if its Tits form is positive definite. In fact, this is true (for any ground field!) and is part of what is known as **Gabriel's Theorem**. The other part of the theorem states that the connected quivers of finite type are exactly those for which the associated undirected graph is a Dynkin diagram of type A, D, or E [**Gab72**]. For these quivers, it turns out that the nonnegative solutions of $q(\mathbf{n}) = 1$, known as the **positive roots**, are precisely the dimension vectors of indecomposable representations, and there is exactly one isomorphism class corresponding to each positive root.

The quiver (253) relevant to the classification of isotropic triples is not of finite type. It does belong, however, to the "next best" class, that of the so-called **tame** quivers. For these, the Tits form is positive *semi*definite, with one-dimensional null space which we will denote by N. N has a smallest positive element, which is the dimension vector of a family of representations whose isomorphism classes are parametrized, in the case of $\mathbf{k}$ algebraically closed, by a 1-dimensional variety. The positive roots thus fall into lines parallel to N. They still correspond to indecomposable representations, which now belong to families indexed by the positive integers.

An extension of Gabriel's theorem (c.f. [**DR76**], [**DF73**]) tells us that a quiver is of (infinite) tame type if and only if the corresponding undirected graph is an **extended Dynkin diagram**; these are obtained from certain Dynkin diagrams by the addition of an edge attached to an extremal vertex. Among these, for instance, is $\widetilde{D_4}$, consisting of four edges attached to a central vertex. If the edges are all oriented to point toward the vertex, the representations of the quiver are closely related to those of the partially ordered set **1+1+1+1** consisting of four incomparable elements. Representations of this poset are just quadruples of subspaces in vector spaces. These were classified by Gel'fand and Ponomarev [**GP72**], who showed that many classification problems in linear algebra reduce to the classification of certain subspace quadruples. For instance, endomorphisms of a vector space V in dimension n correspond to certain kinds of quadruples of subspaces of dimension n in $V \oplus V$ (the "axes," the diagonal, and the graph of the endomorphism).

Indecomposable representations of this kind correspond to indecomposable endomorphisms which, in the case of an algebraically closed field, are those given by a single Jordan block. Since the diagonal element and the size of such a block is arbitrary, one sees immediately the presence of 1-parameter families with arbitrarily large dimension vectors. This example also shows the possible complications for fields which are not algebraically closed, where indecomposable endomorphisms are parametrized by irreducible polynomials which are no longer necessarily linear. We will see later that there are close connections between the representations of $\widetilde{D_4}$ and those of $\widetilde{E_6}$, the latter also being connected to the classification of endomorphisms.

In the next two sections, we reinterpret, via the theory of quiver and poset representations, the easy classifications of symplectic vector spaces and isotropic subspaces, followed by the classification of isotropic pairs, c.f. [**LW15**]. The relevant Dynkin diagrams are A_1, A_3, and A_5.

8.2.3. Symplectic spaces, isotropic subspaces. We start with the classification of symplectic vector spaces with *no* distinguished isotropic subspaces. We may think of these as the symplectic representations of the empty poset, or of the Dynkin diagram A_1, whose quiver consists of a single vertex with no arrows, and whose Tits form in terms of the one-entry dimension vector (v) is the positive definite form v^2. It is well known that any symplectic vector space admits a basis (in fact many such bases) of the form $(e_1, f_1, \dots e_n, f_n)$, with the symplectic form determined by the conditions that $\omega(e_j, f_j) = 1$ for all j and $\omega(a, b) = 0$ for all other pairs (a, b) of basis elements. We call this a **symplectic basis**. As a consequence, any symplectic vector space can be decomposed as a direct sum of copies of the space $\mathbf{k}^2$ with symplectic basis (e, f). The only invariant of a symplectic space is its dimension, which must be an even nonnegative integer. This is consistent with the fact that the Tits form has a single positive root, which is (1).

Another viewpoint here is that there is one indecomposable representation of the empty poset, the 1-dimensional space $\mathbf{k}$. The symplectification of this representation is $\mathbf{k}^* \oplus \mathbf{k}$. In fact, we will use this description of the 2-dimensional symplectic space, rather than $\mathbf{k}^2$.

We now move on to the example of individual isotropic subspaces, which correspond to certain symplectic representations of the poset **2**, i.e. nested pairs of subspaces, as noted above, where isotropic I in V corresponds to the representation $V \supseteq I^\perp \supseteq I$.

The quiver associated to this poset is

$$\bullet \longleftarrow \bullet \longleftarrow \bullet$$

with underlying Dynkin diagram A_3. For a dimension vector of the form (v, c, i), corresponding to a representation $V \leftarrow C \leftarrow I$, the Tits form is $q(v, c, i) = v^2 + c^2 + i^2 - vc - ci$, which can be rewritten as

$$\frac{1}{2}(v^2 + (v - c)^2 + (c - i)^2 + i^2),$$

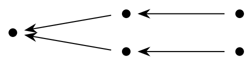
which is clearly positive definite. The dimension vectors of indecomposable representations of the quiver are the positive roots, i.e. nonnegative integer solutions of $q(v, c, i) = 1$. These must be vectors such that exactly two of the four squared summands are equal to 1. Of the six such solutions, those for which the arrows are represented by injective maps are $(1, 0, 0)$, $(1, 1, 0)$, and $(1, 1, 1)$. Over a ground field $\mathbf{k}$, the corresponding representations are $(1; 0, 0) := \mathbf{k} \supseteq 0 \supseteq 0$, $(1; 1, 0) := \mathbf{k} \supseteq \mathbf{k} \supseteq 0$, and $(1; 1, 1) := \mathbf{k} \supseteq \mathbf{k} \supseteq \mathbf{k}$. $(1; 1, 0)$ is self-dual, while $(1; 0, 0)$ and $(1; 1, 1)$ are (isomorphic to) the duals of one another.

On $(1;1,0)$, there are no symplectic forms.[4]

As we have seen in Example 7.6.7, the symplectifications of these representations, contained in $\mathbf{k}^* \oplus \mathbf{k}$, are $(1;1,0)^- = \mathbf{k}^* \oplus \mathbf{k} \supset 0 \oplus 0$, $(1;0,0)^- = \mathbf{k}^* \oplus 0 \supseteq \mathbf{k}^* \oplus 0$, and $(1;1,1)^- = 0 \oplus \mathbf{k} \supseteq 0 \oplus \mathbf{k}$.

The isotropic subspaces in $\mathbf{k}^* \oplus \mathbf{k}$ are the zero subspace in the first case, and (lagrangian) lines in the latter two cases. The latter two symplectic representations are isomorphic, but we will use both of them in the classification of multiple isotropic subspaces.

8.2.4. Isotropic pairs. The relevant poset here is $\mathbf{2}+\mathbf{2}$, to which we associate the quiver



with underlying Dynkin diagram A_5.

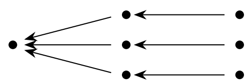
For a representation of this quiver given by maps $I_1 \to C_1 \to V \leftarrow C_2 \leftarrow I_2$, we will write the dimension vector in the form $(v; c_1, i_1; c_2, i_2)$.

The Tits form may be written as

$$q(v; c_1, i_1; c_2, i_2) = \frac{1}{2}[(v-c_1)^2 + (c_1 - i_1)^2 + i_1^2 + (v-c_2)^2 + (c_2-i_2)^2 + i_2^2]$$

Again, this is positive definite, and the positive roots are those vectors making exactly two of the squared terms equal to 1. Those giving representations by injective maps (which can be found by consulting a table of the positive roots of A_5) are as follows. The only self-dual root is $(1;1,0;1,0)$. The others, arranged together with their duals, are $(1;0,0;0,0)$ and $(1;1,1;1,1)$, $(1;1,0;0,0)$ and $(1;1,0;1,1)$, $(1;0,0;1,0)$ and $(1;1,1;1,0)$, and $(1;1,1;0,0)$ and $(1;0,0;1,1)$. Since the self-dual root has an odd-dimensional ambient space, the corresponding representation does not admit a symplectic structure. We then have five indecomposable isotropic pairs by symplectification, all lying in $\mathbf{k}^* \oplus \mathbf{k}$; they are $(0 \oplus \mathbf{k}, 0 \oplus \mathbf{k})$ (two equal lines), $(0 \oplus 0, 0 \oplus \mathbf{k})$ (zero and a line), $(0 \oplus \mathbf{k}, 0 \oplus 0)$ (a line and zero), $(\mathbf{k}^* \oplus 0, 0 \oplus \mathbf{k})$ (two distinct lines), and $(0 \oplus 0, 0 \oplus 0)$ (two zero subspaces). They correspond exactly (in a different order) to the five symplectic indecomposables numbered 6 through 10 in Theorem 2 of [**LW15**].

8.2.5. Overview of indecomposable representations of the poset $\mathbf{2}+\mathbf{2}+\mathbf{2}$. We come now to the central object of this chapter. As noted earlier, the quiver which we associate with the poset $\mathbf{2}+\mathbf{2}+\mathbf{2}$ governing isotropic triples is



and the corresponding extended Dynkin diagram is $\widetilde{E_6}$. An explicit description of the indecomposable $\widetilde{E_6}$ representations has been given by Donovan and Freislich in [**DF73**], organized into families described as follows: The dimension vectors of indecomposables are arranged in lines parallel to $N = \mathbb{N}(3;2,1;2,1;2,1)$. Each of these lines contains a least positive element, followed by vectors obtained by adding the elements of N. These give sequences in increasing dimensions. The dimension vectors in N are referred to

[4]The compatible nondegenerate bilinear forms on $(1;1,0)$ are symmetric; the group $\mathbf{k}^*/(\mathbf{k}^*)^2$, where $\mathbf{k}^*$ is the multiplicative group of $\mathbf{k}$, acts simply and transitively on the isomorphism classes of such forms. This quotient group is $\mathbb{Z}_2$ in the case of a finite field or the real numbers, and the trivial group for an algebraically closed field. It can be much larger, for instance in the case of the rational numbers. In any case, the isotropic subspace is the zero subspace.

as **continuous**, the others as **discrete**; we also use this terminology for the associated indecomposable representations.

REMARK 8.2.2. The classification of discrete-type indecomposables is in fact independent of the ground field $\mathbf{k}$, while the classification of continuous-type indecomposables does (partially) depend on $\mathbf{k}$. This follows from DR [**DR76**], or by inspection of the proofs in DF [**DF73**].

REMARK 8.2.3. In view of Corollary 7.2.1 and the fact that indecomposable discrete-type sextuples are uniquely determined up to isomorphism by their dimension vectors, to show that a sextuple is of a given isomorphism type of discrete type it is sufficient to show that it has the required dimension vector and that its endomorphism ring is local.

An important step in classifying symplectic representations of the poset $P = \mathbf{2}+\mathbf{2}+\mathbf{2}$ with our chosen involution is to identify which linear representations are self-dual, since these may admit compatible symplectic forms. For the discrete-type dimension vectors, self-duality of the dimension-vector implies self-duality of the (uniquely) corresponding indecomposable linear representation, c.f. Lemma 7.5.1. In view of Remark 8.2.2, the self-dual discrete-type sextuples may be read off from the classification in [**DF73**]:

PROPOSITION 8.2.4. *The self-dual discrete dimension vectors of indecomposable representations of* $\mathbf{2} + \mathbf{2} + \mathbf{2}$ *are of the form* $(3k + 1; 2k + 1, k, 2k + 1, k, 2k + 1, k)$ *and* $(3k + 2; 2k + 1, k + 1, 2k + 1, k + 1, 2k + 1, k + 1)$*. For each of these there is, up to isomorphism, a unique indecomposable, named* $A(3k+1, 0)$ *resp.* $A(3k+2, 0)$*. In particular, these are the only self-dual discrete sextuples.*

Explicit descriptions of the isotropic triples associated to the self-dual sextuples $A(3k+1, 0)$ and $A(3k+2, 0)$ are given in Sections 8.3.2 and 8.3.3. In Section 8.3.4 compatible symplectic forms are constructed explicitly, and in Section 8.3.12 we discuss their uniqueness. The (more difficult) question of duality for continuous-type indecomposable sextuples is discussed in Sections 8.4, 8.5, and 8.6.

In the present section we give some further explanations of the DF-classification [**DF73**]. The discrete indecomposable sextuples are labeled in the form $L_s(3k + i, d)$, where k can be any non-negative integer and $i \in \{0, 1, 2, 3\}$ (but does not always run over that whole set). $3k + i$ gives the dimension of the ambient space V, and d is the defect $\sum c_j + \sum i_j - 3v$. L is a letter in $\{A, B, C, D\}$ which encodes the degree of symmetry of the dimension vector, with A = fully symmetric, i.e. all "arms" equal, B or C = exactly two arms equal, D = no arms equal. The subscript s is either empty (when $L = A$), an integer in $\{1, 2, 3\}$ when $L = B$ or C, or a pair of unequal integers in $\{1, 2, 3\}$ (when $L = D$). This subscript encodes "where the asymmetry is". In cases B or C, it tells which arm has a different dimension vector. In case D, it tells how two asymmetric dimension vectors can be related via permutation of the arms. For example, $D_{12}(3k+1, 0)$ is related to $D_{31}(3k + 1, 0)$ via the permutation $1 \to 3$, $2 \to 1$, $3 \to 2$ (but only two indices are needed to specify a permutation of 3 elements).

There is no essential difference between the cases $L = B$ and $L = C$. The reason for using two different letters seems to simply be that, in the defect -1 and $+1$ cases, if the parameters k, i, d, and s are fixed, then there are still two distinct dimension vectors of indecomposables.

The lowest dimensional members of each family of discrete-type indecomposable sextuples are listed in the following table (with $i = 1, 2, 3$):

defect	dim 1	dim 2	dim 3
-3	$A(1,-3)$	$A(2,-3)$	
-2	$C_i(1,-2)$	$C_i(2,-2)$	$C_i(3,-2)$
-1	$B_i(1,-1)$	$B_i(2,-1)$	$B_i(3,-1)$
-1	$C_i(1,-1)$	$C_i(2,-1)$	$C_i(3,-1)$
0	$D_{i+2,i+1}(1,0)$	$D_{i+2,i+1}(2,0)$	
0	$A(1,0)$	$A(2,0)$	
0	$D_{i,i+1}(1,0)$	$D_{i,i+1}(2,0)$	
1	$C_i(1,1)$	$C_i(2,1)$	$C_i(3,1)$
1	$B_i(1,1)$	$B_i(2,1)$	$B_i(3,1)$
2	$C_i(1,2)$	$C_i(2,2)$	$C_i(3,2)$
3	$A(1,3)$	$A(2,3)$	

The row in the middle contains the lowest dimensional members of the two families of discrete indecomposables which are self-dual, namely the families $A(3k+1,0)$ and $A(3k+2,0)$. For any other entry in the table, its dual indecomposable is found by reflecting along the horizontal middle axis, e.g. $A(1,-3)$ and $A(1,3)$ are mutually dual, $C_i(1,-2)$ and $C_i(1,2)$ are mutually dual, etc..

In contrast to the discrete-type indecomposable sextuples, the classification of continuous-type indecomposable sextuples is dependent on the ground field (again, this follows from the classification in DF [**DF73**] and DR [**DR76**]). Although we will ultimately work in the setting where the ground field is only assumed to be perfect, for illustrative purposes, we assume for the moment that the ground field is the complex numbers.

The indecomposable continuous-type sextuples can be arranged into families whose lowest dimensional members are listed in the following table. Here, $i = 1,2,3$, and the parameter λ is understood as ranging in the disjoint union of the sets $\mathbb{C}\backslash\{0,1\}$ and $\{0_i, 1_1, 1_2, \infty_i\}$, where the latter 8 (formal) elements are labels for certain "exceptional" indecomposables.

defect	dim 3
0	$\Delta(1,0_i)$
0	$\Delta(1,\lambda),\ 0<\lvert\lambda\rvert<1,\ \text{or}\ \lvert\lambda\rvert=1\ \text{with}\ \mathrm{Im}\lambda>0$
0	$\Delta(1,1_1)$
0	$\Delta(1,-1)$
0	$\Delta(1,1_2)$
0	$\Delta(1,\lambda^{-1}),\ 0<\lvert\lambda\rvert<1,\ \text{or}\ \lvert\lambda\rvert=1\ \text{with}\ \mathrm{Im}\lambda>0$
0	$\Delta(1,\infty_{i+1})$

As with the previous table, this one is also arranged so that dual sextuples are placed symmetrically to each other with respect to reflection around the central horizontal row (and leaving this row fixed); in particular, the only self-dual element of the table is $\Delta(1,-1)$[5].

[5] We note that, for indecomposable sextuples of the type $\Delta(1,\lambda)$ with $\lambda \in \mathbb{C}\backslash\{0,1\}$, the separation into two families (according to the absolute value of λ and the sign of its imaginary part) is something we have introduced "artificially" for this table in order to emphasize a separation into dual pairs. In fact, the values of λ are not intrinsic; the values used here are simply one of many possible ways to parametrize the "moduli space" of continuous-type indecomposable sextuples.

Finally, we review these results in the context of Section 8.2.2. The quiver we work with, $\tilde{E}_6$, is tame but not of finite type, and its indecomposable representations, and hence those of the underlying poset, are infinite in number and of arbitrarily high dimension, with some of them of discrete type and others of continuous type. Following the pattern in the preceding subsections, we will denote a dimension vector by $(v; c_1, i_1; c_2, i_2; c_3, i_3)$. The Tits form

$$\begin{aligned} q(v; c_1, i_1; c_2, i_2; c_3, i_3) =& \frac{1}{2}[-v^2 + (v - c_1)^2 + (c_1 - i_1)^2 + i_1^2 \\ &+ (v - c_2)^2 + (c_2 - i_2)^2 + i_2^2 + (v - c_3)^2 + (c_3 - i_3)^2 + i_3^2] \end{aligned}$$

is now positive *semi*definite; its null space N is 1-dimensional, generated over the integers by $\nu := (3; 2, 1; 2, 1; 2, 1)$. Dlab-Ringel [**DR76**] give a nice presentation of (a constant times) this form as a sum of six squares. Since there are 7 variables, this shows positive semidefiniteness.

The discrete indecomposable sextuples are those with dimension vectors for which the Tits form takes the value 1.

8.2.6. Triples in dimension 2. We will enumerate here the isotropic triples in dimension 2, which are all symplectically indecomposable. Since many of them are linearly decomposable, we will be using the description in Example 7.6.7 of the symplectifications of the three nested pairs in $\mathbf{k}$.

The following are the possibilities for a triple, in the symplectic plane $\mathbf{k}^* \oplus \mathbf{k}$, of pairs each consisting of a coisotropic subspace containing its isotropic orthogonal. These are necessarily symplectically indecomposable, but only the final example is linearly indecomposable. The rest are symplectifications of 1-dimensional linear representations.

(1) Three copies of $\mathbf{k}^* \oplus \mathbf{k} \supseteq 0 \oplus 0$. (The isotropics are all zero.) This is the symplectification of the sextuple in $\mathbf{k}$ consisting of three copies of $\mathbf{k} \supseteq 0$, whose dimension vector is $(1; 1, 0; 1, 0; 1, 0)$. In the DF classification, this is $A(1, 0)$. It is self-dual, but all compatible bilinear forms are symmetric rather than symplectic.

(2) Two copies of $\mathbf{k}^* \oplus \mathbf{k} \supseteq 0 \oplus 0$ and one copy of $0 \oplus \mathbf{k} \supseteq 0 \oplus \mathbf{k}$. (Two isotropics are zero, and one is a line.) This is the symplectification of two copies of $\mathbf{k} \supseteq 0$ and one copy of $\mathbf{k} \supseteq \mathbf{k}$ (or its dual $0 \supseteq 0$). In terms of the DF classification, the linear representations being symplectified are of the type $B_r(1, 1)$ (or its dual $B_r(1, -1)$), for $r = 1$, 2, or 3. Thus there are three possibilities here, depending upon the value of r (i.e., on which of the three isotropics is a line). The $B_r(1, \pm 1)$ are the first members of the families $B_r(3k + 1, \pm 1)$. $B_3(1, 1)$, for example, has the dimension vector $(1; 1, 0; 1, 0; 1, 1)$, and its dual $B_3(3k + 1, -1)$ has dimension vector $(1; 1, 0; 1, 0; 0, 0)$.

(3) One copy each of $\mathbf{k}^* \oplus \mathbf{k} \supseteq 0 \oplus 0$, $\mathbf{k}^* \oplus 0 \supseteq \mathbf{k}^* \oplus 0$, and $0 \oplus \mathbf{k} \supseteq 0 \oplus \mathbf{k}$. (One isotropic is zero, and the other two are different lines.) This is the symplectification of one copy each of $\mathbf{k} \supseteq 0$, $0 \supseteq 0$, and $\mathbf{k} \supseteq \mathbf{k}$. In terms of the DF classification, the linear representation being symplectified is $D_{12}(1, 0)$, $D_{31}(1, 0)$, or $D_{23}(1, 0)$ (or the dual $D_{32}(1, 0)$, $D_{21}(1, 0)$, or $D_{13}(1, 0)$ respectively). Again, there are three possibilities, depending upon which of the three isotropics is zero. The $D_{ij}(1, 0)$ are the first members of the families $D_{ij}(3k + 1, 0)$.

(4) One copy of $\mathbf{k}^* \oplus \mathbf{k} \supseteq 0 \oplus 0$ and two copies of $0 \oplus \mathbf{k} \supseteq 0 \oplus \mathbf{k}$. (One isotropic is zero, and the other two are identical lines.) This is the symplectification of one copy of $\mathbf{k} \supseteq 0$ and two copies of $\mathbf{k} \supseteq \mathbf{k}$. In the DF classification, this is $C_r(1, 2)$ (or its

dual $C_r(1,-2)$) for $r = 1$, 2, or 3, leading again to three possibilities, depending upon which isotropic is zero. The $C_r(1,\pm 2)$ are the first members of the families $C_r(3k+1,\pm 2)$.

(5) Three copies of $0 \oplus \mathbf{k} \supseteq 0 \oplus \mathbf{k}$. (All three isotropics are the same line.) This is the symplectification of three copies of $\mathbf{k} \supseteq \mathbf{k}$ (or its dual $0 \supseteq 0$). In the DF classification, this is $A(1,3)$ (or $A(1,-3)$). The corresponding dimension vector is $(1;1,1;1,1;1,1)$ (or $(1;0,0;0,0;0,0)$). The $A(1,\pm 3)$ are the first members of the families $A(3k+1,\pm 3)$.

(6) Two copies of $0 \oplus \mathbf{k} \supseteq 0 \oplus \mathbf{k}$ and one copy of $\mathbf{k}^* \oplus 0 \supseteq \mathbf{k}^* \oplus 0$. (Two of the isotropics are the same line, and the third one is a different line.) This is the symplectification of two copies of $\mathbf{k} \supseteq \mathbf{k}$ and one copy of $0 \supseteq 0$ (or vice versa). In the DF classification, this is $C_r(1,1)$ (or its dual $C_r(1,-1)$), for $r = 1$, 2, or 3. Again we have three possibilities, depending upon which line is distinct from the other two. The $C_r(1,\pm 1)$ are the first members of the families $C_r(3k+1,\pm 1)$.

(7) The final case, that of three distinct lines in a plane, is linearly indecomposable. In the DF classification, as a linear representation, it is $A(2,0)$, which is self-dual. Its dimension vector is $(2;1,1;1,1;1,1)$. The number of isomorphism classes of symplectic representations with this underlying linear representation depends on the ground field $\mathbf{k}$. We may find a symplectic basis (e_1, f_1) whose elements span I_1 and I_2, respectively. I_3 is then spanned by $e_1 + af_1$ for some nonzero $a \in \mathbf{k}$. If we change the basis to $be_1, b^{-1}f_1$, then I_3 is spanned by $be_1 + baf_1 = be_1 + b^2a(b^{-1}f_1)$. This implies that the set of isomorphism classes of triples of lines may be parametrized (taking the case $a = 1$ as "basepoint") by the square class group $\mathbf{k}^\times/\mathbf{k}^{\times 2}$ introduced in Remark 7.8.2. Thus, when $\mathbf{k}$ is algebraically closed, there is just one isomorphism class of this type, while in the case $\mathbf{k} = \mathbb{R}$, there are two.[6] In this case, the isomorphism class is invariant only under cyclic permutations of the three lines, and the **Maslov index** of the triple distinguishes the two possibilities[7]. $A(2,0)$ is the first member of the family $A(3k+2,0)$ whose members for k even admit compatible symplectic structures. Those for odd k require symplectification.

We conclude that the number of isomorphism classes of isotropic triples in dimension 2 is $1+3+3+3+1+3+\#(\mathbf{k}^\times/\mathbf{k}^{\times 2})$, or $14 + \#(\mathbf{k}^\times/\mathbf{k}^{\times 2})$, where the last term is the order of the square class group.

8.2.7. Higher dimensions: A preview. All of the isotropic triples in dimension 2 were listed in the previous subsection. Again, they are the symplectifications, for the case $k = 0$, of $A(3k+1,0)$, $B_r(3k+1,\pm 1)$, $D_{ij}(3k+1,0)$, $C_r(3k+1,\pm 2)$, $A(3k+1,\pm 3)$, $C_r(3k+1,\pm 1)$, along with the isotropic triples which arise from compatible forms for the self-dual sextuple $A(2,0)$.

For higher k, the members of the family $A(3k+2,0)$ are always self-dual (see Section 8.3 below), and they admit symplectic forms if only if k is even. Similarly, the members of the family $A(3k+1,0)$ are all self-dual and admit symplectic forms if and only if k is

[6]For information about the square class group of other fields, we refer to [**Bec01**], [**Lam99**], and [**Raj93**]. For instance, the square class group of a finite field has order 1 or 2 according to whether the characteristic is even (i.e. 2) or odd. For the p-adic numbers, the order of the square class group is 8 for p = 2, and 4 otherwise.

[7]Here we are referring to the Maslov index for Lagrangian triples, also known as the Kashiwara index, see [**Dui76**], [**LV80**].

odd. Thus, for $k = 1$ or $k = 2$ one finds non-split indecomposable isotropic triples arising from compatible forms for $A(3+1,0)$ and $A(6+2,0)$, respectively. These are in ambient dimension 4 and 8, respectively. The symplectifications of $A(3+2,0)$ and $A(6+1,0)$ give indecomposable isotropic triples in ambient dimension 10 and 16, respectively.

In dimension 4, beside the non-split isotropic triples associated with $A(3+1,0)$, we have, for $k = 0$, the symplectifications of $A(3k+2,\pm 3)$, $C_r(3k+2,\pm 2)$, $B_r(3k+2,\pm 1)$, $C_r(3k+2,\pm 1)$, and $D_{ij}(3k+2,0)$.

In dimension 6, we have, for $k = 0$, the symplectifications of the discrete sextuples $C_r(3k+3,\pm 2)$, $B_r(3k+1,\pm 1)$, and $C_r(3k+3,\pm 1)$, and for $k = 1$ the symplectifications of the non self-dual continuous-type indecomposable sextuples. In addition, for $k = 2$ there are the non-split isotropic triples arising from compatible symplectic forms for the self-dual continuous-type sextuples. As we will see later in this chapter, non-split continuous-type isotropic triples pose the most intricate case to study, and the number of such isotropic triples depends in particular also on the ground field.

We end our preview of the "higher landscape" of isotropic triples with the remark that indecomposable isotropic triples exist in every even dimension. This follows from the fact that there exist indecomposable non-self-dual sextuples in every given dimension; their symplectifications therefore give indecomposable isotropic triples in every even dimension.

8.2.8. Hamiltonian vector fields. In this section we outline briefly how another problem in linear symplectic geometry can be treated using symplectic poset representations of **1+1+1+1**, and how this problem can in turn be encoded in isotropic triples. As a result, we will get our first example of isotropic triple of continuous type.

Set $P =$ **1+1+1+1** and consider the involution $\perp$ on P which exchanges the first two elements and leaves the last two elements fixed. This involution is trivially order-reversing, since all elements of P are incomparable. Symplectic poset representations of $(P,\perp)$ are then subspace systems $(V; U_1, U_2, U_3, U_4)$ where V is symplectic, U_1 and U_2 are mutually orthogonal, and U_3 and U_4 are lagrangian.

The problem we will discuss is that of classifying linear hamiltonian vector fields up to conjugation by linear symplectomorphisms; in other words, the classification of the orbits of the Lie algebra $\mathrm{sp}(V,\omega)$ under the adjoint action of the symplectic group $\mathrm{Sp}(V,\omega)$. This is a problem whose solution is well-known and has a long history (going back to Williamson [**Wil37**] in the 1930s). It has since been treated by various authors, in particular with a view toward finding special normal forms adapted to applications; see, for example, [**Koc84**], [**LM74**].

Let (V,ω) be a symplectic vector space. A linear hamiltonian vector field X on V is an element of the Lie algebra $\mathfrak{sp}(V,\omega)$; i.e. it is a linear map $X : V \to V$ such that $\tilde{\omega} \circ X = -X^* \circ \tilde{\omega}$. One wishes to understand equivalence classes, where one linear hamiltonian vector field (V_1,ω_1,X_1) is equivalent to another, (V_2,ω_2,X_2), if there exists a linear symplectomorphism $\phi : V_1 \to V_2$ such that $X_2 \circ \phi = \phi \circ X_1$. There is a natural notion of direct sum, and any linear hamiltonian vector field is the direct some of indecomposable pieces. We wish, here, to point out how one may view linear hamiltonian vector fields as symplectic poset representations of $(P,\perp)$. For this we proceed in two steps: first, in the following lemma, we reformulate hamiltonian vector fields in terms of certain kinds of subspaces.

Lemma 8.2.5. *There is a bijective correspondence between linear hamiltonian vector fields (V, ω, X) and linear maps $f : V \to V^*$ such that graph$(f) \subseteq V \times V^*$ is a symplectic subspace with respect to the canonical symplectic form on $V \times V^*$.*

Proof. We give only a sketch. Given (V, ω, X), it is readily checked that the graph of $f_X := (\tilde{\omega} \circ X) + \tilde{\omega}$ is a symplectic subspace.

Conversely, if $f : V \to V^*$ is a linear map whose graph is a symplectic subspace, then the asymmetric part f_a will be invertible, and hence defines a symplectic structure ω_f on V. Setting $X_f := f_a^{-1} f_s$, one finds that X_f is hamiltonian with respect to ω_f.

It is straightforward to check that the two operations are mutually inverse to one another. □

Corollary 8.2.6. *A linear hamiltonian vector field (V, ω, X) can be encoded in the symplectic representation of $(P, \perp)$*

$$(V \times V^*; \mathit{graph}(f_X), \mathit{graph}(f_X)^{\perp}, V \times 0, 0 \times V^*).$$

Although we do not show it here, the passage from a linear hamiltonian vector field to the associated symplectic representation of $(P, \perp)$ is functorial and compatible with the respective notions of direct sum.

Next we show how the objects above can be encoded in isotropic triples. Observe that the symplectic poset representation of $(P, \perp)$ which we associated to a linear hamiltonian vector field is such that the first two subspaces, which are mutually orthogonal, are symplectic subspaces; in other words, they are independent to each other. The last two subspaces, which are lagrangian, are also independent, and all four subspaces have the same dimension. In the following we will consider only those symplectic poset representations of $(P, \perp)$ which are of this kind.

Given such a symplectic representation $\varphi = (V; S, S^{\perp}, L_1, L_2)$, let ω_V denote the symplectic form on V, ω_S the restriction of ω_V to the symplectic subspace S, and let $\overline{S}$ denote a copy of S equipped with the symplectic form $-\omega_S$. From φ we construct the following isotropic triple in the ambient symplectic space $V \times \overline{S}$, with form $\omega := \omega_V \times -\omega_S$:

$$\begin{array}{ll} I_1 = L_1 \times 0 & C_1 = L_1 \times \overline{S} \\ I_2 = L_2 \times 0 & C_2 = L_2 \times \overline{S} \\ I_3 = \{(x, x) \mid x \in S\} & C_3 = I_3 + (S^{\perp} \times 0) \end{array} \tag{254}$$

The passage from φ to (254) is also functorial and compatible with direct sums. Thus, combining the above, we obtain a way of turning any linear hamiltonian vector field into an isotropic triple. We will not analyze in full detail here exactly which types of isotropic triples can be built from linear hamiltonian vector fields. The following, though, describes a large class of isotropic triples which can.

Proposition 8.2.7. *Let $\varphi = (V; C_i, I_i)$ be an isotropic triple such that*

(1) $V = I_1 \oplus I_2 \oplus I_3$, *with* $\dim I_i = 1/3 \dim V$ *for* $i = 1, 2, 3$,

(2) $I_i + I_j$ *is a symplectic subspace for all* $i \neq j$.

Then we can construct from φ a symplectic form f_a and a linear hamiltonian vector field X on I_2 such that φ is isomorphic to the isotropic triple (254) obtained from (I_2, f_a, X).

Remark 8.2.8. Since the geometric description of the isotropic triples in this proposition is invariant under permutation of the indices, the choice of I_2 for carrying the hamiltonian vector field is arbitrary. We make this particular choice in order to have coherence with certain types of normal forms which we will use later.

PROOF. We can write V as the sum $S \oplus S'$ of $I_1 \oplus I_2$ and its symplectic orthogonal. I_1 and I_2 being a lagrangian decomposition of S, we can identify I_1 with I_2^* and, hence, S with the "cotangent bundle" T^*I_2.

Now I_3 cannot intersect S', or its sums with I_1 and I_2 would not be symplectic subspaces. So I_3 is the graph of a map $g : S \leftarrow S'$ which is antipresymplectic, since I_3 is isotropic. This means that this map g pulls back the symplectic form on S to the negative of that on S'. This implies that g is injective and that its image $g(S')$ is a symplectic subspace of S which is independent of I_1 and I_2. Thus, $g(S')$ is the graph of a map $f : I_2 \to I_2^*$, which can be considered as a bilinear form on I_2. Since $g(S')$ is symplectic rather than isotropic, the form is not symmetric; in fact, it has a nondegenerate antisymmetric part f_a. Writing $f = f_a + f_s$ as the sum of its antisymmetric and symmetric parts, since f_a is invertible, we can form the product $f_a^{-1} f_s$, which is a linear hamiltonian vector field on the symplectic space (I_2, f_a).

To see that the isotropic triple φ is isomorphic to the one of the form (254) associated to (I_2, f_a, X), observe that g defines a symplectomorphism $\overline{S} \leftarrow S'$. It then easy to check that the direct sum of g with the 'identity map' on $S = I_1 \oplus I_2$ defines a symplectomorphism from φ to (254). □

Let us look at an example, to see that isotropic triples of the kind in Proposition 8.2.7 do exist. In fact the indecomposable ones come in families dependent on a parameter taking a continuum of values. Indeed, the isomorphism class of the isotropic triple (254) built from a linear hamiltonian vector field X depends on X up to conjugation of X by symplectomorphisms, so the spectrum of X is an invariant of the isotropic triple.

EXAMPLE 8.2.9. Let $V = \mathbb{R}^6$ with symplectic basis $(f_1, f_2, f_3, e_1, e_2, e_3)$. Set $I_1 =< f_3, e_1 >, I_2 =< f_1, e_3 >$, and $I_3 =< -\lambda f_1 + (\lambda - 1) f_2 + f_3, e_1 +_2 +e_3 >$, and let λ vary. Then some computation shows that $X = \tilde{\omega}^{-1}\tilde{\sigma}$ as above is given by the matrix whose two diagonal entries are $1 - \lambda$ and $1 + \lambda$, so the associated triples for different values of λ are non-isomorphic and give a nontrivial 1-parameter family.

REMARK 8.2.10. In ambient dimension 6, it is easy to see directly that isotropic triples of the kind in Proposition 8.2.7 are symplectically indecomposable. If there were a decomposition, it would be an orthogonal splitting of the form $\mathbf{k}^2 \oplus \mathbf{k}^4$. Looking at the possible ways in which each of the isotropics decomposes, it is not hard to check that the induced 2 form on one of the sums must have rank 2 rather than 4, a contradiction.

8.3. Discrete non-split isotropic triples

Having given an overview of some background material in the representation theory of posets and quivers, and its connection to isotropic triples, we begin now with the details of our classification. In this section, we study those sextuples which are self-dual and of discrete type, and how they may give rise to non-split isotropic triples. Discrete-type sextuples are somewhat simpler to study than the continuous-type sextuples; the latter are studied in the subsequent, remaining sections of this chapter (except for the last section).

Recall that the indecomposable discrete-type sextuples are uniquely characterized by their dimension vector. In particular, self-dual indecomposable discrete-type sextuples are precisely those whose dimension vector is self-dual, which means here that $c_j + i_j = v$ for $j = 1, 2, 3$[8]. Thus, such self-dual sextuples may be read off from the classification in

[8]See Lemma 7.5.1.

[**DF73**]. As stated already in Proposition 8.2.4, the discrete indecomposable sextuples with self-dual dimension vector are denoted $A(3k+i,0)$, for $k \in Z_+$ and i equal to 1 or 2. The dimension vectors are

$$(3k+1; 2k+1, k; 2k+1, k; 2k+1, k) \qquad \text{for } A(3k+1,0), \text{ and}$$
$$(3k+2; 2k+1, k+1; 2k+1, k+1; 2k+1, k+1) \qquad \text{for } A(3k+2,0).$$

By Lemmas 7.5.1 and 7.5.3, each of these representations admits a compatible form. The degree of uniqueness of such forms is specified in Theorem 8.3.12 below. Of course, for the symplectic case, non-split istropic triples can only arise in cases of k odd for $A(3k+1,0)$ and k even for $A(3k+2,0)$, since only then is the ambient vector space V even-dimensional. In fact, whenever V *is* even-dimensional, the compatible forms granted by Lemma 7.5.3 *are* symplectic; this follows from Theorem 8.3.3 and Theorem 8.3.12. The self-dual discrete-type sextuples having odd ambient dimension, on the other hand, lead to isotropic triples via symplectification.

8.3.1. Small dimensions. We give here brief geometric descriptions of the lowest-dimensional non-split discrete-type isotropic triples. These follow, for example, from the normal forms given in the subsections below.

In the previous section, we already saw the first example of a non-split isotropic triple: the underlying sextuple is of type $A(2,0)$ (it belongs to the $A(3k+2,0)$-family), consisting of three distinct lines in a plane.

Next, the non-split isotropic triple arising from the sextuple $A(4,0)$, which belongs to the $A(3k+1,0)$-family. Here, the dimension vector is $(4;3,1;3,1;3,1)$. The three smaller subspaces (in this case they are lines) are independent, and for each line, the 3-dimensional subspace containing it is independent of the other two lines. With a compatible symplectic form, the isotropic triple we obtain has the following form: the isotropic subspaces are three lines I_i in a 4-dimensional space V, and each of their orthogonal subspaces C_i is independent from the other two isotropics. Furthermore, $(C_i \cap C_j) \cap (I_i + I_j) = 0$ for any $i \neq j$, so any two of the isotropics span a symplectic subspace. Since the three isotropics are independent, their sum C has codimension 1 and is hence coisotropic. Its orthogonal $C^\perp = C_1 \cap C_2 \cap C_3 \subseteq I_1 + I_2 + I_3$ is a line which must be pairwise independent with each of the I_i: if not, then $C^\perp = I_i$ would be the case for some i, and hence $C = C_i$ and so C_i would contain all the istropics, a contradiction. Thus the I_i and $C^\perp$ are four lines in general position in V. Symplectic reduction via the coisotropic C gives a non-split isotropic triple of the type $A(2,0)$.

Moving on, the next case is the sextuple $A(8,0)$, which is in the $A(3k+2,0)$-family. We find the dimension vector to be $(8;5,3;5,3;5,3)$, so the corresponding isotropics are a triple of 3-spaces I_i in an 8 dimensional symplectic space V. Though the isotropics I_i are pairwise independent, they are not fully independent: for each distinct triple of indices, $Q_i := I_i \cap (I_j + I_k)$ is a line. The three lines Q_1, Q_2, Q_3 are themselves pairwise-independent, and span the 2-dimensional space

$$I = (I_1 + I_2) \cap (I_2 + I_3) \cap (I_3 + I_1).$$

This space I is contained in its orthogonal,

$$C = (C_1 \cap C_2) + (C_2 \cap C_3) + (C_3 \cap C_1)$$

and is hence isotropic. Symplectic reduction by the coisotropic C gives a non-split isotropic triple of the type $A(4,0)$.

We consider one more case. The underlying sextuple is of type $A(10,0)$, which is in the $A(3k+1,0)$-family. The dimension vector is $(10;7,3;7,3;7,3)$. Thus we are again dealing with 3-dimensional isotropics, but this time in a 10-dimensional ambient space. As in the 4-dimensional example above, the isotropics here are completely independent, and their sum is a codimension 1 subspace C which is therefore coisotropic. Its orthogonal,

$$C^{\perp} = (I_1 + I_2 + I_3)^{\perp} = C_1 \cap C_2 \cap C_3$$

is a line which is pairwise independent with each of the I_i. Symplectic reduction via C gives a non-split isotropic triple of the type $A(8,0)$.

8.3.2. Implementing the $A(3k+1,0)$. Given $k \geq 0$ and a basis

$$\beta = (e_1, \dots e_{k+1}, f_1, \dots, f_k, g_1, \dots, g_k)$$

of a vector space V^β, a sextuple $(V^\beta; C_i^\beta, I_i^\beta)$ of type $A(3k+1,0)$ is given by

$$\begin{aligned} I_1^\beta &= \langle e_1 - f_1, ..., e_k - f_k\rangle, \\ I_2^\beta &= \langle g_1, ..., g_k\rangle, \\ I_3^\beta &= \langle f_1 - g_1, ..., f_k - g_k\rangle, \\ \\ C_1^\beta &= \langle e_1, ..., e_{k+1}, f_1, ..., f_k\rangle \\ C_2^\beta &= \langle e_1, ..., e_{k+1}, g_1, ..., g_k\rangle \\ C_3^\beta &= \langle e_1, e_2 - f_1, ..., e_{k+1} - f_k, f_1 - g_1, ..., f_k - g_k\rangle. \end{aligned} \tag{255}$$

Note that

$$\begin{aligned} C_1^\beta &= I_1^\beta + \langle e_1, ..., e_{k+1}\rangle \\ C_2^\beta &= I_2^\beta + \langle e_1, ..., e_{k+1}\rangle \\ C_3^\beta &= I_3^\beta + \langle e_1, e_2 - f_1, ..., e_{k+1} - f_k\rangle = I_3^\beta + \langle e_1, e_2 - g_1, ..., e_{k+1} - g_k\rangle. \end{aligned}$$

In view of Remark 8.2.3, to show that this really does define an isotropic triple of type $A(3k+1,0)$ it suffices to observe that the above sextuple has the required dimension vector

$$\dim V^\beta = 3k+1, \quad \dim I_i^\beta = k, \quad \dim C_i^\beta = 2k+1$$

and that its endomorphism algebra is local. The latter follows from Lemma 7.3.3 and the following.

LEMMA 8.3.1. *Let ψ be an indecomposable sextuple of type $A(3k+1,0)$. Then*

$$End(\psi) \simeq End((U,\eta))$$

where η is an indecomposable nilpotent endomorphism and $\dim U = k+1$. *In particular, $End(\psi)$ is local and $End(\psi) = \mathbf{k}id \oplus Rad$.*

PROOF. Let ψ be given in the normal form (255). Consider the End(ψ)-invariant subspace

$$U := C_1^\beta \cap C_2^\beta = \langle e_1, ..., e_{k+1}\rangle$$

and the indecomposable nilpotent endomorphism η of U defined by $\eta(e_1) = 0$ and $\eta(e_{i+1}) = e_i$, for $i = 2, ..., k$. The endomorphism algebra of (U,η), i.e. the algebra of endomorphisms of U which commute with η, is local because η is indecomposable[9]. We will see now that this algebra is isomorphic to the endomorphism algebra of the sextuple (255), hence the

[9]See Lemma 7.3.3. There it is also noted that this endomorphisms algebra E is such that $E = \mathbf{k}\mathrm{id} \oplus \mathrm{Rad}E$.

latter is local as well. To do this, we use the following map. Given an endomorphism a of U which commutes with η, we can extend it to an endomorphism $\overline{a}$ of (255) by defining linear isomorphisms

$$f : \langle e_1, ..., e_k\rangle \to \langle f_1, ..., f_k\rangle, \qquad f(e_i) := f_i \quad \forall i = 1, ..., k,$$
$$g : \langle f_1, ..., f_k\rangle \to \langle g_1, ..., g_k\rangle, \qquad g(f_i) := g_i \quad \forall i = 1, ..., k,$$

and setting

$$\begin{aligned} \bar{a} &:= faf^{-1} && \text{on } \langle f_1, ..., f_k\rangle, \\ \bar{a} &:= g\bar{a}g^{-1} && \text{on } \langle g_1, ..., g_k\rangle. \end{aligned} \tag{256}$$

(Note that domain of f is invariant under a.)

Since $\overline{a}$ is defined via its action on the basis β, it is easily checked directly that $\overline{a}$ is an endomorphism of (255). To see this, note in particular that $I_1^\beta = \{x - f(x) \mid x \in \langle f_1, ..., f_k\rangle\}$ and $I_3^\beta = \{x - g(x) \mid x \in \langle g_1, ..., g_k\rangle\}$.

The map $a \mapsto \overline{a}$ has as its inverse the operation of taking an endomorphism b of (255) and restricting it to $U = C_1^\beta \cap C_2^\beta$ (which will necessarily be an invariant subspace of b, since by assumption C_1^β and C_2^β are b-invariant). To see this, notice that such a b necessarily decomposes as the direct sum of its restrictions to the subspaces

$$U = \langle e_1, ..., e_{k+1}\rangle, \quad F := \langle f_1, ..., f_k\rangle, \quad G := \langle g_1, ..., g_k\rangle$$

since these subspaces sum to V and must be invariant under b:

$$U = C_1^\beta \cap C_2^\beta, \quad F = I_2^\beta, \quad G = C_1^\beta \cap (I_1^\beta + I_2^\beta).$$

The invariance of I_1^β and I_3^β under b then enforces that $b_{|U}$ is related to $b_{|F}$ and $b_{|G}$ via (256), and together with the invariance of $C_3^\beta \cap C_1^\beta$ it is ensured that $b_{|U}$ commutes with η.

Thus if we restrict b to U and then extend to $\overline{b}$, we recover b. Conversely, if we start with an endomorphism a of U, the restriction of $\overline{a}$ to U is of course, by definition, again a.

Finally, it is clear from (256) that the operation $a \mapsto \overline{a}$ is a morphism of algebras. □

8.3.3. Implementing the $A(3k+2, 0)$. We will in fact work with $A(3(k-1)+2, 0) = A(3k-1, 0)$, which is a subquotient of $A(3k+1, 0)$ for $k > 0$, so that the construction of compatible forms for sextuples of both types $A(3k+1, 0)$ and $A(3k+2, 0)$ can be treated uniformly.

Given $k \geq 1$ and a basis $\gamma = (e_2, \dots e_k, f_1, \dots, f_k, g_1, \dots, g_k)$ of a vector space V^γ, a sextuple $(V^\gamma; C_i^\gamma, I_i^\gamma)$ of type $A(3k-1, 0)$ is given by

$$\begin{aligned} I_1^\gamma &= \langle f_1, f_2 - e_2, ..., f_k - e_k\rangle \\ I_2^\gamma &= \langle g_1, ..., g_k\rangle, \\ I_3^\gamma &= \langle f_1 - g_1, ..., f_k - g_k\rangle, \\ \\ C_1^\gamma &= \langle e_2, ..., e_k, f_1, ..., f_k\rangle \\ C_2^\gamma &= \langle e_2, ..., e_k, g_1, ..., g_k\rangle \\ C_3^\gamma &= \langle f_1 - e_2, ..., f_{k-1} - e_k, f_1 - g_1, ..., f_k - g_k\rangle. \end{aligned} \tag{257}$$

Note that

$$\begin{aligned} C_1^\gamma &= I_1^\gamma + \langle e_2, ..., e_k\rangle = I_1^\gamma + \langle f_2, ..., f_k\rangle \\ C_2^\gamma &= I_2^\gamma + \langle e_2, ..., e_k\rangle \\ C_3^\gamma &= I_3^\gamma + \langle f_1 - e_2, ..., f_{k-1} - e_k\rangle. \end{aligned}$$

Again, in view of Remark 8.2.3, it suffices to observe that such a sextuple has the required dimension vector

$$\dim V^\gamma = 3k-1, \quad \dim I_i^\gamma = k, \quad \dim C_i^\gamma = 2k-1$$

and local endomorphism algebra. For the latter, we proceed similarly as for sextuples of type $A(3k+1,0)$, combining Lemma 7.3.3 and the following.

LEMMA 8.3.2. *Let ψ be an indecomposable sextuple of type $A(3k-1,0)$. Then*

$$End(\psi) \simeq End((U,\eta))$$

where η is an indecomposable nilpotent endomorphism and $\dim U = k$. *In particular, $End(\psi)$ is local and $End(\psi) = \mathbf{k}id \oplus Rad$.*

PROOF. Let ψ be given in the normal form (257). Consider the End(ψ)-invariant subspace

$$U := \langle f_1, ..., f_k\rangle = C_1^\gamma \cap (I_2^\gamma + I_3^\gamma)$$

and the indecomposable nilpotent endomorphism η of U defined by $\eta(f_1) = 0$ and $\eta(f_{i+1}) = f_i$, for $i = 2, ..., k$. Similarly as in the previous section, the algebra of endomorphisms of U which commute with η is isomorphic to the endomorphism algebra of (257).

To show this, we begin with an endomorphism a of U which commutes with η and extend it to an endomorphism $\overline{a}$ of V^γ by defining maps f, g, and h by

$$\begin{array}{lll} f : \langle e_2, ..., e_k\rangle \to \langle f_1, ..., f_k\rangle, & f(e_i) = f_i \text{ for } i = 2, .., k, \\ g : \langle f_1, ..., f_k\rangle \to \langle g_1, ..., g_k\rangle, & g(f_i) = g_i \text{ for } i = 1, .., k, \\ h : \langle f_1, ..., f_k\rangle \to \langle e_2, ..., e_k\rangle, & h(f_1) = 0, \quad h(f_i) = e_i \text{ for } i = 2, .., k. \end{array}$$

and setting

$$(258) \qquad \begin{array}{ll} \overline{a} := haf & \text{on } \langle e_2, ..., e_k\rangle. \\ \overline{a} := gag^{-1} & \text{on } \langle g_1, ..., g_k\rangle. \end{array}$$

Note that $I_1^\gamma = \{x - h(x) \mid x \in \langle f_1, ..., f_k\rangle\}$ and $I_3^\gamma = \{x - g(x) \mid x \in \langle f_1, ..., f_k\rangle\}$, and that

$$E = \langle e_2, ..., e_k\rangle, \quad U = \langle f_1, ..., f_k\rangle, \quad G = \langle g_1, ..., g_k\rangle$$

are subspaces invariant under the endomorphism algebra of (257) since

$$E = C_1^\gamma \cap C_2^\gamma, \quad U = C_1^\gamma \cap (I_2^\gamma + I_3^\gamma), \quad G = I_2^\gamma.$$

For an endomorphism b of (257), the operation $b \mapsto b_{|F^\gamma}$ is inverse to the operation $a \mapsto \overline{a}$. Indeed, such a b decomposes as the direct sum $b = b_{|E} \oplus b_{|U} \oplus b_{|G}$, and the relations (258) are enforced by the invariance of I_1^γ and I_3^γ under b.

Finally, that the map $a \mapsto \overline{a}$ is a morphism of algebras is evident from (258). □

The bases β and γ are referred to as **standard bases** for the respective types of sextuple. For each fixed $k > 0$ one can view (257) as a subquotient of (255) by considering the subspaces $I^\beta \subseteq C^\beta \subseteq V^\beta$,

$$I^\beta = \langle e_1\rangle, \quad C^\beta = \langle e_1, ..., e_k, f_1, ..., f_k, g_1, ..., g_k\rangle,$$

and letting $V^\gamma = C^\beta/I^\beta$, where the basis γ is induced (modulo re-indexing) by the elements of the basis β which span C (i.e. all elements except e_{k+1}). Then we have the identifications

$$I_i^\gamma = (I_i^\beta \cap C^\beta)/I^\beta \quad \text{and} \quad C_i^\gamma = (C_i^\beta \cap C^\beta)/I^\beta \quad \text{for } i = 1, 2, 3.$$

In fact, we can also view a sextuple of type $A(3(k-2)+1,0) = A(3k-5,0)$ as a subquotient of a sextuple of type $A(3k-1,0)$. If a sextuple of the latter type is given in the form (257), then choosing

$$I^\gamma = \langle f_1, g_1\rangle, \quad C^\gamma = \langle e_2, ..., e_k, f_1, ..., f_{k-1}, f_1, ..., g_{k-1}\rangle,$$

and passing to the subquotient C^γ/I^γ gives a sextuple of type $A(3(k-2)+1,0)$, with standard basis induced from the standard basis of (257).

In total, the (isomorphism classes of) discrete non-split sextuples form two "chains of subquotients" (arrows indicate the passage to a subquotient):

	$k=0$	$k=1$	$k=2$	$k=3$	$\dots$
$A(3k+1,0)$	$A(1,0)$	$A(4,0)$	$A(7,0)$	$A(10,0)$	$\dots$
$A(3k+2,0)$	$A(2,0)$	$A(5,0)$	$A(8,0)$	$A(11,0)$	$\dots$

8.3.4. Existence of compatible forms.

THEOREM 8.3.3. *Any sextuple of type $A(3k\pm 1,0)$ admits ε-symmetric forms where $\varepsilon = (-1)^k$. With respect to standard bases there is a recursion providing compatible ε-symmetric forms with coefficients in the prime subfield.*

The existence will be shown by constructing matrices for these compatible forms with respect to standard bases. These matrices will be of the shape (259), i.e. having a block structure and built using a (smaller) matrix which we call "A".

Fix $k \in \mathbb{N}$, let $\varepsilon = (-1)^k$, and let $A = A_k \in \mathbf{k}^{(k+1)\times(k+1)}$. We consider the following relations on the entries of A:

CONDITIONS 8.3.4.

(1) $a_{ij} = 0$ for $i+j < k+2$, $\quad a_{ij} \neq 0$ for $i+j = k+2$
(2) $A = \varepsilon A^t$
(3) $a_{i,j+1} = a_{ij} - a_{i+1,j}$ for $i,j = 1,\dots,k$
(4) $a_{kk} = a_{k,k+1} + a_{k+1,k}$.

Further we define $\mathring{A} = \mathring{A}_k$ as the minor of A given by restricting to row and column indices in $\{2,\dots,k\}$ and we set

$$c_i = a_{i,k+1}, \quad c_i' = a_{k+1,i}.$$

If (1) of Conditions 8.3.4 holds, then the matrix A has the following form

$$A = \left[\begin{array}{c|ccccccc|c}
0 & 0 & 0 & \cdots & \cdots & 0 & 0 & a_{1,k+1} \\ \hline
0 & 0 & 0 & \cdots & \cdots & 0 & a_{2,k} & a_{2,k+1} \\
0 & 0 & 0 & \cdots & \cdots & a_{3,k-1} & a_{3,k} & a_{3,k+1} \\
\vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots \\
\vdots & \vdots & 0 & & & \vdots & \vdots & \vdots \\
0 & 0 & a_{k-1,3} & \cdots & \cdots & a_{k-1,k-1} & a_{k-1,k} & a_{k-1,k+1} \\
0 & a_{k2} & a_{k3} & \cdots & \cdots & a_{k,k-1} & a_{kk} & a_{k,k+1} \\ \hline
a_{k+1,1} & a_{k+1,2} & a_{k+1,3} & \cdots & \cdots & a_{k+1,,k-1} & a_{k+1,k} & a_{k+1,k+1}
\end{array}\right].$$

The vertical and horizontal lines inside of this matrix are intended solely as visual aids.

Now we define

$$
H = H_k := \left[\begin{array}{c|ccc|c||c|ccc||ccc|c}
0 & 0 & \cdots & 0 & c_1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \cline{2-13}
0 & & & & c_2 & 0 & & & & & & & 0 \\
\vdots & & \mathring{A} & & \vdots & \vdots & & \mathring{A} & & & 0 & & \vdots \\
0 & & & & c_k & 0 & & & & & & & 0 \\ \hline
c_1' & c_2' & \cdots & c_k' & c_{k+1} & c_1' & c_2' & \cdots & c_k' & 0 & \cdots & 0 & 0 \\ \hline\hline
0 & 0 & \cdots & 0 & c_1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & c_1 \\ \cline{2-13}
0 & & & & c_2 & 0 & & & & & & & c_2 \\
\vdots & & \mathring{A} & & \vdots & \vdots & & \mathring{A} & & & \mathring{A} & & \vdots \\
0 & & & & c_k & 0 & & & & & & & c_k \\ \hline\hline
0 & & & & 0 & 0 & & & & & & & 0 \\
\vdots & & 0 & & \vdots & \vdots & & \mathring{A} & & & 0 & & \vdots \\
0 & & & & 0 & 0 & & & & & & & 0 \\ \hline
0 & 0 & \cdots & 0 & 0 & c_1' & c_2' & \cdots & c_k' & 0 & \cdots & 0 & 0
\end{array}\right] \tag{259}
$$

and we interpret this matrix to be the coordinate matrix of a bilinear form B on V^β (c.f. Section 8.3.2) with respect to the standard basis $\beta = \{e_1, ..., e_{k+1}, f_1, ..., f_k, g_1, ..., g_k\}$. The double lines in the matrix are visual aids for seeing the block structure

$$
H = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix}
$$

related to the subspaces $\langle e_1, ..., e_{k+1}\rangle$, $\langle f_1, ..., f_k\rangle$, and $\langle g_1, ..., g_k\rangle$.

CLAIM 8.3.5. If (1) and (2) of Conditions 8.3.4 hold, then B is non-degenerate and ε-symmetric

PROOF. That B is ε-symmetric follows directly from (**??**). To see that B is non-degenerate, note that the blocks of the matrix H are such that H_{33}, H_{31} and H_{13} are zero and where H_{11}, H_{23}, and H_{32} are square matrices having non-zero entries on the anti-diagonal, and zeros above the anti-diagonal; in particular, the latter blocks are invertible. Because H_{13} is zero and because a 'copy' of H_{21} is contained in H_11, we can use row operations to turn transform H_{21} to zero in such a way that only H_{22} is additionally changed under these operations. In a similarly manner we can also turn H_{12} to zero using column operations. At this point our block matrix has the following form (tilde indicates that there are changes)

$$
\tilde{H} = \begin{bmatrix} H_{11} & 0 & 0 \\ 0 & \tilde{H}_{22} & H_{23} \\ 0 & H_{32} & 0 \end{bmatrix}
$$

and where $\tilde{H}_{22}$ is

$$
\left[\begin{array}{c|ccc}
0 & 0 & \cdots & 0 \\ \hline
0 & & & \\
\vdots & & \mathring{A} & \\
0 & & &
\end{array}\right]
-
\left[\begin{array}{c|ccc}
0 & 0 & \cdots & 0 \\ \hline
0 & & & \\
\vdots & & \mathring{A} & \\
0 & & &
\end{array}\right]
-
\left[\begin{array}{c|ccc}
0 & 0 & \cdots & 0 \\ \hline
0 & & & \\
\vdots & & \mathring{A} & \\
0 & & &
\end{array}\right]
= -
\left[\begin{array}{c|ccc}
0 & 0 & \cdots & 0 \\ \hline
0 & & & \\
\vdots & & \mathring{A} & \\
0 & & &
\end{array}\right].
$$

Now, clearly we can add columns from the third block column to the second block column to transform $\tilde{H}_{22}$ into the zero matrix, and this leaves all other blocks unchanged, since

H_{13} and H_{33} are zero. (Equivalently, we could have used row transformations using rows from the bottom block row.) This puts our block matrix in the form

$$\begin{bmatrix} H_{11} & 0 & 0 \\ 0 & 0 & H_{23} \\ 0 & H_{32} & 0 \end{bmatrix}$$

which shows non-degeneracy, since the non-zero blocks are non-degenerate. □

CLAIM 8.3.6. If (2) of Conditions 8.3.4 holds, then

$$B(I_1^\beta, C_1^\beta) = B(I_2^\beta, C_2^\beta) = B(I_3^\beta, \langle e_1, e_2 - g_1, ..., e_{k+1} - g_k\rangle) = 0.$$

PROOF.
- $B(e_i,\ e_j - f_j) = a_{ij} - a_{ij} = 0$ for $i = 1, \dots, k+1$ and $j = 1, \dots, k$.
- $B(f_i,\ e_j - f_j) = a_{ij} - a_{ij} = 0$ for $i, j = 1, \dots, k$.
- $B(e_i, g_j) = 0$ for $i = 1, \dots, k+1,\ j = 1, \dots, k$. $B(g_i, g_j) = 0$ for $i, j = 1, \dots, k$. $B(e_1,\ f_j - g_j) = B(e_1, f_j) - B(e_i, g_j) = 0 - 0 = 0$ for $i, j = 1, \dots, k$.
- $B(f_i - g_i,\ e_{j+1} - g_j) = B(f_i, e_{j+1}) - B(f_i, g_j) - 0 + 0 = a_{i,j+1} - a_{i,j+1} = 0$ for $i, j = 1, \dots, k$.

□

CLAIM 8.3.7. If Conditions 8.3.4 hold, then B is ε-symmetric and a compatible form for the sextuple $(V^\beta; C_i^\beta, I_i^\beta)$.

PROOF. It remains to show $B(I_3^\beta, C_3^\beta) = 0$. Consider $B(f_i - g_i,\ f_j - g_j) = B(f_i, f_j) - B(f_i, g_j) - B(g_i, f_j) + B(g_i, g_j) =: x_{ij}$. Direct checking gives:
- $x_{11} = 0 + 0 + 0 + 0 = 0$.
- $x_{1j} = 0 + 0 + a_{2j} + 0 = 0$ for $j = 2, \dots, k-1$.
- $x_{ij} = a_{ij} - a_{i,j+1} - a_{i+1,j} + 0 = 0$ for $i, j = 2, \dots, k-1$ by (3).
- $x_{ik} = a_{ik} - c_i - a_{i+1,k} + 0 = 0$ for $i = 2, \dots, k-1$ by (3).
- $x_{1k} = 0 - c_1 - a_{2k} + 0 = 0$ by (3).
- $x_{kk} = a_{kk} - c_k - c'_k = 0$ by (4).

In view of Claim 8.3.6 and that

$$I_3^\beta = \langle f_1 - g_1, ..., f_k - g_k\rangle$$
$$C_3^\beta = \langle e_1, e_2 - f_1, ..., e_{k+1} - f_k, f_1 - g_1, ..., f_k - g_k\rangle$$

we have compatibility of B. □

CLAIM 8.3.8. For each $k = 0, 1, 2, ...$ there exist matrices A_k satisfying (**??**)–(**??**).

PROOF. We proceed by induction on k. Let

$$A_0 = (c_1), \quad A_1 = \begin{pmatrix} 0 & c_1 \\ -c_1 & 0 \end{pmatrix}, \quad c_1 \neq 0$$

Assume that A_{k-2} is given satisfying Conditions 8.3.4. We define $A = A_k$ to have minor $\mathring{A} = A_{k-2}$ with respect to row and column indices $2, \dots, k$. Thus, we have

(1') for $2 \leq i, j \leq k$: $a_{ij} = 0$ if $i + j < k + 2$, $a_{ij} \neq 0$ if $i + j = k + 2$
(2') $\mathring{A} = \varepsilon \mathring{A}^t$
(3') $a_{i,j+1} = a_{ij} - a_{i+1,j}$ for $i, j = 2, \dots, k-1$,
(4') $a_{k-1,k-1} = a_{k-1,k} + a_{k,k-1}$.

It remains to grant

(1") $a_{1j} = a_{i1} = 0$ for $i, j = 1, \dots, k$ and $a_{1,k+1} \neq 0$, $a_{k+1,1} \neq 0$

(2") $a_{k+1,i} = \varepsilon a_{i,k+1}$ for $i = 1, \ldots, k+1$
(3") $a_{i,j+1} = a_{ij} - a_{i+1,j}$ for $j = k$, $i = 1, \ldots$ resp. $i = k$, $j = 1, \ldots, k-1$
(4") $a_{kk} = a_{k,k+1} + a_{k+1,k}$.

Define

$$\begin{array}{rcll} a_{1,k+1} & = & -a_{2,k} & \\ a_{i,k+1} & = & a_{i,k} - a_{i+1,k} & \text{for } i = 2, \ldots, k-1 \\ a_{k,k+1} & = & \frac{1}{2} a_{k,k} & \text{if } k \text{ is even} \\ a_{k,k+1} & = & \text{arbitrary} & \text{if } k \text{ is odd} \\ a_{k+1,k+1} & = & \text{arbitrary} & \text{if } k \text{ is even} \\ a_{k+1,k+1} & = & 0 & \text{if } k \text{ is odd} \\ a_{k+1,i} & = & \varepsilon a_{i,k+1} & \text{for } i = 1, \ldots, k. \end{array}$$

Then (1") and (2") are obvious, as is (3") for the cases when $j = k$, while for those cases with $i = k$ we have $a_{k,j+1} = \varepsilon a_{j+1,k} = \varepsilon(a_{j,k} - a_{j,k+1}) = a_{kj} - a_{k+1,j}$. Finally, if k is odd then $a_{kk} = \varepsilon a_{kk} = 0 = a_{k,k+1} + \varepsilon a_{k,k+1} = a_{k,k+1} + a_{k+1,k}$; and if k is even then $a_{kk} = \frac{1}{2} a_{kk} + \frac{1}{2} a_{kk} = a_{k,k+1} + a_{k+1,k}$. □

Proof of Theorem 8.3.3. For the $A(3k+1, 0)$, Claims 8.3.7 and 8.3.8 and the proof of the latter give the recursive construction of such forms with respect to standard bases. For $A(3k-1, 0)$, we view V^γ as a subspace of V^β and define on it a form given by a matrix $H' = H'_k$ obtained from $H = H_k$ by omitting all rows and columns indexed by e_1 or e_{k+1}. Then H' defines an ε-symmetric form B' on V^γ. To see this, let $S^\beta := \langle e_1, e_{k+1} \rangle$ and note that $H_{|_{S^\beta}}$ is non-degenerate and ε-symmetric, and that we can make the identification $V^\gamma = (S^\beta)^\perp \subseteq V^\beta$. Hence $H' = H_{|_{(S^\beta)^\perp}}$ is also non-degenerate and ε-symmetric. It remains to show that H' is compatible with the $A(3k-1, 0)$ sextuple in V^γ obtained from the $A(3k+1, 0)$ sextuple in V^β.

Recall that if $X \subseteq V^\beta$ is an element of the sextuple in V^β, then the corresponding subspace of the sextuple in V^γ is given by $\underline{X} := \pi(X^\beta \cap C^\beta)$, where π denotes the projection onto the second factor of the (orthogonal) decomposition $V^\beta = S^\beta \oplus V^\gamma$. Since $C^\beta = (C^\beta)^\perp \oplus V^\gamma$ is coisotropic, this is an instance of coisotropic reduction; in particular

$$B'(\underline{X}, \underline{Y}) = B(X \cap C^\beta, Y \cap C^\beta)$$

for any $X, Y \subseteq V^\beta$. Hence $B(X, Y) = \{0\}$ implies that $B'(\underline{X}, \underline{Y}) = \{0\}$. Since we know the dimensions of all the subspaces involved in our sextuples, this shows that for any element X of the sextuple in V^β, the orthogonal of $\underline{X}$ in V^γ is the same subspace as $\underline{X^\perp}$. □

Remark 8.3.9. Suppose we are given sextuples of the types $A(3k+1, 0)$ and $A(3k-1, 0)$ in terms of standard bases.

(1) With respect to the standard basis, any compatible form on $A(3k+1, 0)$ is given by a matrix H_k which is built from a matrix A as above, and such that A satisfies Conditions 8.3.4. The analogous statement is true for compatible forms on $A(3k-1, 0)$ in terms of matrices of the form H'_k, where H'_k is obtained from H_k by deleting the 1st and $(k+1)th$ rows and columns.
(2) A parametrization of such matrices H_k resp. H'_k is given by the parameters

$$a_{k+1,j}, \quad j = 1, \ldots, k+1 \qquad \text{resp.} \qquad a_{k,j}, \quad j = 2, \ldots, k$$

Moreover, the matrix entries are obtained via linear expressions from the parameters.

PROOF. (i) According to the proofs of Claims 8.3.6 and 8.3.7, the structure of B and Conditions 8.3.4 are forced by the requirement of admissibility.

(ii) This follows from the recursive construction of the matrices A_k; in each step one free parameter can be chosen.

□

EXAMPLE 8.3.10. Let $k = 1$. A sextuple of type $A(3k+1, 0)$ has ambient dimension 4. To construct a compatible symplectic form (with respect to a standard basis) we begin with the matrix

$$A_1 = \begin{pmatrix} 0 & c_1 \\ -c_1 & 0 \end{pmatrix}$$

where c_1 is any non-zero scalar. From this we obtain the matrix (259) of a compatible symplectic form; in this example it is

$$H_1 = \left(\begin{array}{cc|c|c} 0 & c_1 & 0 & 0 \\ -c_1 & 0 & -c_1 & 0 \\ \hline 0 & c_1 & 0 & c_1 \\ \hline 0 & 0 & -c_1 & 0 \end{array}\right)$$

(note that $\mathring{A}_1$ is the empty matrix). To obtain a compatible symplectic form for the sextuple $A(3k-1, 0)$, with respect to a standard basis induced from $A(3k+1, 0)$, we only need to drop the $1st$ and $(k+1)th$ rows and columns of the compatible form H_1 given above. This gives

$$H_1' = \begin{pmatrix} 0 & c_1 \\ -c_1 & 0 \end{pmatrix}.$$

EXAMPLE 8.3.11. Let $k = 3$. A sextuple of type $A(3k+1, 0)$ has ambient dimension 10. To construct a compatible symplectic form (with respect to a standard basis) we proceed similarly as in the previous example. Following the recursion recipe given in Claim 8.3.8 above, we build from A_1 the matrix

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & c_1 \\ 0 & 0 & -c_1 & -c_1 \\ 0 & c_1 & 0 & c_2 \\ -c_1 & c_1 & -c_2 & 0 \end{pmatrix}$$

where c_2 is a scalar that we may freely choose. Now from A_3 we obtain the matrix (259) of a compatible symplectic form; in this example it is

$$H_3 = \left(\begin{array}{cccc|ccc|ccc} 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -c_1 & -c_1 & 0 & 0 & -c_1 & 0 & 0 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_1 & 0 & 0 & 0 & 0 \\ -c_1 & c_1 & -c_2 & 0 & -c_1 & c_1 & -c_2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 & 0 & c_1 \\ 0 & 0 & -c_1 & -c_1 & 0 & 0 & c_1 & 0 & -c_1 & -c_1 \\ 0 & c_1 & 0 & c_2 & 0 & -c_1 & 0 & c_1 & 0 & c_2 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -c_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -c_1 & c_1 & -c_2 & 0 & 0 & 0 \end{array}\right).$$

To obtain a compatible symplectic form for the sextuple $A(3k-1,0)$, we drop the $1st$ and $(k+1)th$ rows and columns of H_3. This gives

$$H_3' = \left(\begin{array}{cc|ccc|ccc} 0 & -c_1 & 0 & 0 & -c_1 & 0 & 0 & 0 \\ c_1 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 \\ 0 & -c_1 & 0 & 0 & c_1 & 0 & -c_1 & -c_1 \\ c_1 & 0 & 0 & -c_1 & 0 & c_1 & 0 & c_2 \\ \hline 0 & 0 & 0 & 0 & -c_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -c_1 & c_1 & -c_2 & 0 & 0 & 0 \end{array}\right).$$

8.3.5. Uniqueness of compatible forms.

THEOREM 8.3.12. *The $(-1)^k$-symmetric forms on an $A(3k\pm1,0)$ are unique up to isometric automorphism and multiplication by scalars; there are no $(-1)^{k+1}$-symmetric forms on an $A(3k\pm1,0)$.*

If B is a compatible ε-symmetric form on $A(3k\pm1,0)$ and $c\in\mathbf{k}$, then there is an automorphism η of the sextuple which is an isometry, in the sense $\eta^*B=cB$, if and only if c is a square in $\mathbf{k}$.

PROOF. Dealing with $A(3k+1,0)$, we continue the discussion from Subsection 7.8. By Lemma 8.3.1, we may apply Lemma 7.8.1; this yields the first claim.

Now, suppose there exists an automorphism η of $A(3k+1,0)$ such that $f^*B=cB$, and let η_U to be its restriction to $U=C_1^\beta\cap C_2^\beta$, which is invariant under η. We may assume that B is given by a matrix B_k in terms of a standard basis as above. Thus, $U=\langle e_1,...,e_{k+1}\rangle$ is non-degenerate and the restriction of B to U has matrix A which is zero above the anti-diagonal. By Observation 7.3.3, η_U has upper triangular matrix (η_{ij}) with diagonal entries all the same. It follows that

$$\begin{aligned} cB(e_{k+1},e_1) &= B(\eta e_{k+1},\eta e_1) = B(\sum_{i=1}^{k+1}\eta_{i,k+1}e_i,\eta_{11}e_1) \\ &= B(\eta_{k+1,k+1}e_{k+1},\eta_{11}e_1) = B(\eta_{11}e_{k+1},\eta_{11}e_1) = \eta_{11}^2 B(e_{k+1},e_1) \end{aligned}$$

whence $c=\eta_{11}^2$. (Conversely, given a square $c=b^2\in\mathbf{k}$, one can of course always find an isometry between B and cB: simply take $b\cdot\mathrm{id}$.)

For $k>1$, a similar reasoning works for $A(3k-1,0)$, with $U=C_1^\gamma\cap C_2^\gamma=\langle e_2,...,e_k\rangle$. □

8.4. Continuous types: sextuple classification and duality

By definition, continuous-type indecomposable sextuples are those indecomposable sextuples which have dimension vectors of the form $(3k;2k,k;2k,k;2k,k)$, for integers $k\geq1$. Note that this is a self-dual dimension vector.

As mentioned above, and proven in [**DF73**, Section 4.6], when $\mathbf{k}$ is algebraically closed the (isomorphism types of) continuous-type indecomposable sextuples in a given dimension (indicated by k) consist of:

- a collection $\Delta(k;\lambda)$, labeled by scalars $\lambda\in\mathbf{k}\backslash\{0,1\}$,

- so-called **exceptional** continuous-type sextuples, which are labeled

$$\begin{aligned}&\Delta_j(k,1) \text{ for } j\in\{1,2\},\\&\Delta_j(k,0) \text{ for } j\in\{1,2,3\},\\&\Delta_j(k,\infty), \text{ for } j\in\{1,2,3\}.\end{aligned}$$

In the case of more general fields, the continuous-type indecomposable sextuples $\Delta(k;\lambda)$ must be replaced by indecomposable sextuples $\Delta(k;\gamma)$, where γ ranges over all indecomposable endomorphisms of $\mathbf{k}^k$ which do not have 0 or 1 as an eigenvalue. The exceptional continuous-type indecomposable sextuples remain the same. Thus the isomorphism classes of continuous-type indecomposable sextuples in a given dimension $3k$ are parametrized by the disjoint union of the sets

$$\{\gamma \mid \gamma \text{ indecomposable endomorphism of } \mathbf{k}^k, \text{ with } 0,1\notin \operatorname{spec}(\gamma)\}$$

and

$$\{0_1,0_2,0_3,1_1,1_2,\infty_1,\infty_2,\infty_3\},$$

where the latter set consists of formal labels for the exceptional types.

We review the classification and structure of continuous-type sextuples in Subsection 8.4.1 below. To further analyze continuous-type sextuples, we recall the notion of a frame in Subsection 8.4.2 and use these in Subsection 8.4.3 to build continuous-type sextuples from underlying linear endomorphisms. In Subsection 8.4.4 we give a detailed description of morphisms of such sextuples, and in Subsection 8.4.5 we identify which indecomposable continuous-type sextuples are dual to which. This sets the stage for the analysis of self-dual continuous indecomposable sextuples and their compatible forms in Sections 8.5 and 8.6.

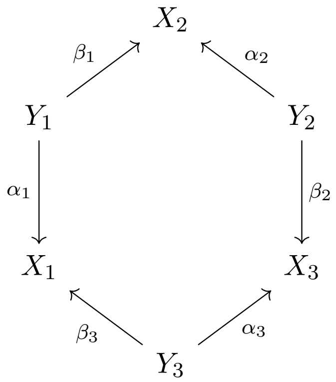
Naturally, when $\mathbf{k}$ is not algebraically closed, the classification of indecomposable sextuples and associated isotropic triples becomes more complicated. First of all, the indecomposable endomorphisms γ underlying the sextuples have a richer structure. Furthermore, self-dual continuous-type indecomposable sextuples may admit more compatible forms when $\mathbf{k}$ is not algebraically closed. The general question of uniqueness of compatible forms is treated in Subsection 8.6.5.

As always, if we are looking for compatible symplectic structures, we can restrict our attention to the cases where the integer k is even.

8.4.1. Classification of continuous sextuples. With the sole exception being the type $\Delta_2(k;1)$, all indecomposable continuous-type sextuples (up to isomorphism) may be obtained via a functor S from the "continuous-type" indecomposable representations of the extended Dynkin quiver $\tilde{A}_5$ (with a certain orientation, as reflected in the diagram

(260))[10]. Indecomposable representations of this $\tilde{A}_5$ quiver have the following normal form

(260)



where $X_i = Y_i = \mathbf{k}^k$ for some $k \in \mathbb{Z}_{>0}$, and $i \in \{1, 2, 3\}$, and the linear maps α_i and β_i are all the identity map on $\mathbf{k}^k$, with one exception. The map which is an exception – we call it γ – is a linear endomorphism of $\mathbf{k}^k$, and the representation of $\tilde{A}_5$ is indecomposable if and only γ is an indecomposable linear endomorphism. The following are the continuous-type indecomposable representations of $\tilde{A}_5$, up to isomorphism (in each case, k runs over $\mathbb{Z}_{>0}$):

- $\Xi(k; \gamma)$: γ runs over isomorphism classes of automorphisms of $\mathbf{k}^k$ and we assume $\gamma = \beta_1$. Any other choice of "position" of γ, e.g. $\gamma = \alpha_1$, leads to a representation which is isomorphic to $\Xi(k; \gamma')$ for some $\gamma' = \beta_1$.
- $\Xi_i(k; 0)$ and $\Xi_i(k; \infty)$: γ is the unique indecomposable nilpotent endomorphism on $\mathbf{k}^k$ in Jordan normal form (for nilpotent γ this normal form always exists), and $i \in \{1, 2, 3\}$. The convention is that $\Xi_i(k; 0)$ denotes the case when $\gamma = \beta_i$ and $\Xi_i(k; \infty)$ denotes the case when $\gamma = \alpha_i$.

Although DF [**DF73**] work over algebraically closed fields, their classification of the indecomposable representations of $\tilde{A}_5$ does not depend on this, and admits the straightforward generalization above, where single Jordan blocks are replaced with the condition of indecomposability. Our notation is a slight modification of their notation.

The image under S of a continuous-type $\tilde{A}_5$ representation is the sextuple

(261) $$V = X_1 \oplus X_2 \oplus X_3, \quad C_i = X_i \oplus X_{i+1}, \quad I_i = \mathrm{Im}\,(\alpha_i \times \beta_i),$$

where indices are understood modulo 3. It will be convenient for us to cast these normal forms in slightly different notation. We set $X = \mathbf{k}^k$,

$$X_1 = X \times 0 \times 0, \; X_2 = 0 \times X \times 0, \; X_3 = 0 \times 0 \times X,$$

and $V = X \times X \times X$. We call an endomorphism **exceptional** if it has 0 or 1 as eigenvalue. Otherwise, it is **non-exceptional**. Note that any direct summand of a non-exceptional endomorphism is again non-exceptional.

For non-exceptional γ, the sextuples which are isomorphic to $S\Xi(k; -\gamma)$[11] will be called of type $\Delta(k; \gamma)$. Normal forms for these sextuples are

(262) $$\begin{array}{ll} I_1 = \{(x, -\gamma x, 0) \mid x \in X\} & C_1 = X \times X \times 0 \\ I_2 = \{(0, x, x) \mid x \in X\} & C_2 = 0 \times X \times X \\ I_3 = \{(x, 0, x) \mid x \in X\} & C_3 = X \times 0 \times X. \end{array}$$

[10] See [**DF73**], pages 44 and 46

[11] The change of sign in front of γ here follows the conventions of DF [**DF73**].

For indecomposable γ having eigenvalue 1, the sextuple $S\Xi(k;-\gamma)$ is called of type $\Delta_1(k;1)$; its normal form is the same as above. The only continuous-type indecomposable sextuple not isomorphic to a sextuple in the image of the functor S is the type $\Delta_2(k;1)$. It is obtained from $\Delta_1(k;1)$ via certain functor θ^+ (see [**DF73**], p. 38 and 46); a normal form for $\Delta_2(k;1)$ is:

$$\begin{array}{ll} I_1 = \{(0,0,x) \mid x \in X\} & C_1 = \{(y, \gamma y, x) \mid x, y \in X\} \\ I_2 = \{(x,0,0) \mid x \in X\} & C_2 = \{(x, y, -y) \mid x, y \in X\} \\ I_3 = \{(0,x,0) \mid x \in X\} & C_3 = \{(-y, x, y) \mid x, y \in X\}, \end{array} \tag{263}$$

where γ is indecomposable and with eigenvalue 1. Finally, for the cases when γ is nilpotent we set $\Delta_i(k;0) := S\Xi_i(k;0)$ and $\Delta_i(k;\infty) := S\Xi_i(k;\infty)$, for $i = 1,2,3$. For normal forms we take the same spaces C_1, C_2, C_3 as in (262), and

- for $\Delta_1(k;0)$ and $\Delta_1(k;\infty)$, respectively:

$$\begin{array}{lcl} I_1 = \{(x, \gamma x, 0) \mid x \in X\} & & I_1 = \{(\gamma x, x, 0) \mid x \in X\} \\ I_2 = \{(0, x, x) \mid x \in X\} & \text{and} & I_2 = \{(0, x, x) \mid x \in X\} \\ I_3 = \{(x, 0, x) \mid x \in X\} & & I_3 = \{(x, 0, x) \mid x \in X\} \end{array}$$

- for $\Delta_2(k;0)$ and $\Delta_2(k;\infty)$:

$$\begin{array}{lcl} I_1 = \{(x, x, 0) \mid x \in X\} & & I_1 = \{(x, x, 0) \mid x \in X\} \\ I_2 = \{(0, x, \gamma x) \mid x \in X\} & \text{and} & I_2 = \{(0, \gamma x, x) \mid x \in X\} \\ I_3 = \{(x, 0, x) \mid x \in X\} & & I_3 = \{(x, 0, x) \mid x \in X\} \end{array}$$

- for $\Delta_3(k;0)$ and $\Delta_3(k;\infty)$:

$$\begin{array}{lcl} I_1 = \{(x, x, 0) \mid x \in X\} & & I_1 = \{(x, x, 0) \mid x \in X\} \\ I_2 = \{(0, x, x) \mid x \in X\} & \text{and} & I_2 = \{(0, x, x) \mid x \in X\} \\ I_3 = \{(\gamma x, 0, x) \mid x \in X\} & & I_3 = \{(x, 0, \gamma x) \mid x \in X\}. \end{array}$$

The following will be useful for identifying isomorphism types.

LEMMA 8.4.1. *Let $(V; I_i, C_i)$ be an indecomposable continuous-type sextuple with* $\dim V = 3k$. *Consider the following eight subspaces: $I_1 + I_2 + I_2$ and $C_1 \cap C_2 \cap C_3$, $I_1 \cap C_3$ and $I_3 \cap C_1$, $I_2 \cap C_1$ and $I_1 \cap C_2$, $I_3 \cap C_2$ and $I_2 \cap C_3$. Let $\epsilon = (\epsilon_1, ..., \epsilon_8)$ be the corresponding 8-vector of the dimensions of these spaces. The different possible types of indecomposable continuous-type sextuple have the following associated 8-vectors ϵ.*

(1) $\Delta(k;\gamma)$, *then* $\epsilon = (3k, 0, 0, 0, 0, 0, 0, 0)$
(2) $\Delta_1(k;1)$, *then* $\epsilon = (3k-1, 0, 0, 0, 0, 0, 0, 0)$
(3) $\Delta_2(k;1)$, *then* $\epsilon = (3k, 1, 0, 0, 0, 0, 0, 0)$
(4) $\Delta_1(k;0)$, *then* $\epsilon = (3k, 0, 1, 0, 0, 0, 0, 0)$
(5) $\Delta_3(k;\infty)$, *then* $\epsilon = (3k, 0, 0, 1, 0, 0, 0, 0)$
(6) $\Delta_2(k;0)$, *then* $\epsilon = (3k, 0, 0, 0, 1, 0, 0, 0)$
(7) $\Delta_1(k;\infty)$, *then* $\epsilon = (3k, 0, 0, 0, 0, 1, 0, 0)$
(8) $\Delta_3(k;0)$, *then* $\epsilon = (3k, 0, 0, 0, 0, 0, 1, 0)$
(9) $\Delta_2(k;\infty)$, *then* $\epsilon = (3k, 0, 0, 0, 0, 0, 0, 1)$

PROOF. Consider first the sextuples for which γ is an isomorphism. It is straightforward to check, e.g. using the normal forms above, that for such sextuples $\dim I_j \cap C_l = 0$ for all $j \neq l$. Thus ϵ_2 through ϵ_8 are zero for the types $\Delta(k;\gamma)$, $\Delta_1(k;1)$ and $\Delta_2(k;1)$.

Furthermore, if a sextuple is of type $\Delta(k;\gamma)$ or $\Delta_1(k;1)$, then from the normal form (262) we see that $C_1 \cap C_2 \cap C_3 = 0$ and that

$$(x,y,z) \in I_1 \cap (I_2 + I_3) \;\Leftrightarrow\; (x,y,z) = (x, -\gamma x, 0) \text{ with } x - \gamma x = 0,$$

so $I_1 \cap (I_2 + I_3) \neq 0$ if and only if γ has 1 as eigenvalue. In the case $\Delta_1(k;1)$ when γ does have 1 as eigenvalue, the corresponding eigenspace has dimension 1 (because γ is indecomposable) and so $\dim I_1 \cap (I_2 + I_3) = 1$. Thus in this case

$$\dim(I_1 + I_2 + I_3) = \dim I_1 + \dim(I_2 + I_3) - 1 = 3k - 1 = \dim V - 1.$$

So, we have found that $(\epsilon_1, \epsilon_2) = (3k, 0)$ for $\Delta(k;\gamma)$ and $(\epsilon_1, \epsilon_2) = (3k-1, 0)$ for $\Delta_1(k;1)$.

For sextuples of type $\Delta_2(k;1)$, it follows from the normal form (263) that $I_1 \cap (I_2 + I_3) = 0$ and that

$$C_1 \cap C_2 \cap C_3 = \{(x, \gamma x, -\gamma x) \mid x = \gamma x\}.$$

Since for the type $\Delta_2(k;1)$ the map γ has a 1-dimensional eigenspace for the eigenvalue 1, we find that $\dim C_1 \cap C_2 \cap C_3 = 1$. So, in this case $(\epsilon_1, \epsilon_2) = (3k, 1)$.

Now consider the type $\Delta_1(k;0)$. The same arguments as for the case $\Delta(k;\gamma)$ show here that $(\epsilon_1, \epsilon_2) = (3k, 0)$. Note that

$$I_1 \cap C_3 = \{(x, \gamma x, 0) \in I_1 \mid \gamma x = 0\} = \ker \gamma,$$

which is 1-dimensional since γ is an indecomposable nilpotent map. From the normal form for $\Delta_1(k;0)$ is easily check that the other intersections $I_j \cap C_l$, $j \neq l$, are zero. Thus $\epsilon_3 = 1$, and ϵ_4 through ϵ_8 are zero.

The remaining cases are very similar to the case $\Delta_1(k;0)$ and may be treated analogously. □

COROLLARY 8.4.2. *Suppose we are given an indecomposable continuous-type sextuple with ambient dimension* $3k$. *The sextuple is of type*

1. $\Delta(k;\gamma)$ *if and only if* $\dim(I_1 + I_2 + I_2) = \dim V$ *and* $\dim(C_1 \cap C_2 \cap C_3) = 0$.
2. $\Delta_1(k;1)$ *if and only if* $\dim(I_1 + I_2 + I_2) = \dim V - 1$
3. $\Delta_2(k;1)$ *if and only if* $\dim(C_1 \cap C_2 \cap C_3) = 1$
4. $\Delta_i(k;0)$ *if and only if* $\dim(I_i \cap C_{i-1}) = 1$.
5. $\Delta_i(k;\infty)$ *if and only if* $\dim(I_i \cap C_{i+1}) = 1$.

8.4.2. Frames. Following von Neumann [**vN98**], we introduce an abstract kind of coordinate system. Given a vector space V, a **frame** for V is a collection of five subspaces $A_1, A_2, A_3, A_{12}, A_{23}$ satisfying the following relations:

$$V = A_1 \oplus A_2 \oplus A_3 \tag{264}$$

$$A_1 + A_2 = A_1 \oplus A_{12} = A_2 \oplus A_{12} \tag{265}$$

$$A_2 + A_3 = A_2 \oplus A_{23} = A_3 \oplus A_{23} \tag{266}$$

As a shorthand notation, we refer to a frame as $\bar{A}$. The notions of morphism and isomorphism of frames are the obvious ones (i.e. view a frame as a special kind of poset representation.)

The following shows that the definition could also be phrased in a way that is more symmetrical.

LEMMA 8.4.3. *Suppose we are given a frame* $A_1, A_2, A_3, A_{12}, A_{23} \subseteq V$. *We define* A_{31} *by*

$$A_{31} = (A_3 + A_1) \cap (A_{12} + A_{23}). \tag{267}$$

Then

$$A_3 + A_1 = A_3 \oplus A_{31} = A_1 \oplus A_{31}. \tag{268}$$

PROOF. To see that $A_3 \cap A_{31} = 0$, note that $\dim V = 3n$ for some $n \in \mathbb{N}$, and

$$A_3 \cap A_{31} = A_3 \cap [(A_1 + A_3) \cap (A_{12} + A_{23})] = A_3 \cap (A_{12} + A_{23}).$$

Thus

$$\dim(A_{31} \cap A_3) = \dim A_3 + \dim(A_{12} + A_{23}) - \dim(A_3 + A_{12} + A_{23}) = 3n - 3n = 0$$

since, via (264), (265), (266), and (267),

$$A_{12} \cap A_{23} = 0 \quad \text{and} \quad A_3 + A_{12} + A_{23} = A_3 + A_{12} + A_2 = A_3 + A_1 + A_2 = V.$$

Similar dimension arguments can be used to show that $A_1 \cap A_{31} = 0$ and that $\dim A_{31} = n$. □

COROLLARY 8.4.4. *If $A_1, A_2, A_3, A_{12}, A_{23} \subseteq V$ is a frame, and A_{31} defined as above, then*

$$A_{12} = (A_1 + A_2) \cap (A_{23} + A_{31}), \tag{269}$$

$$A_{23} = (A_2 + A_3) \cap (A_{31} + A_{12}). \tag{270}$$

PROOF. If $\{A_1, A_2, A_3, A_{12}, A_{23}\}$ is a frame, then by Lemma 8.4.3 also $\{A_1, A_2, A_3, A_{23}, A_{31}\}$ is a frame. Application of Lemma 8.4.3 to this latter frame gives (269); an analogous argument gives (270). □

The relations (264), (265), (266), and (268) imply that $\dim A_1 = \dim A_2 = \dim A_3 = 1/3 \dim V$ and that each A_{ij} can be interpreted as the negative[12] graph of a linear isomorphism $h_{ij} : A_i \to A_j$, i.e.

$$A_{ij} = \{x - h_{ij}(x) \mid x \in A_i\},$$

where $ij \in \{12, 23, 31\}$. We set $h_{ii} := id$ and $h_{ji} := h_{ij}^{-1}$.

LEMMA 8.4.5.

(1) *Given a frame in V, the associated maps satisfy $h_{jk} \circ h_{ij} = h_{ik}$ for any indices $i, j, k \in \{1, 2, 3\}$.*
(2) *Any frame is isomorphic to one built from a vector space U in the following way:*

$$\begin{array}{ll} V = U \times U \times U & \\ A_1 = U \times 0 \times 0 & A_{12} = \{(x, -x, 0) \mid x \in U\} \\ A_2 = 0 \times U \times 0 & A_{23} = \{(0, x, -x) \mid x \in U\} \\ A_3 = 0 \times 0 \times U & A_{31} = \{(x, 0, -x) \mid x \in U\} \end{array}$$

PROOF. (1) Since we are dealing only with invertible maps, equations of the form $h_{jk} \circ h_{ij} = h_{ik}$ are equivalent to ones obtained by applying, to both sides of an equation, the operations of inversion, or pre- or post-composition with one of the "h" maps. This allows one to reduce to the case of showing a single identity, say

$$h_{23} \circ h_{12} = h_{13}. \tag{271}$$

[12] Following von Neumann, we use negative graphs for symmetry reasons when dealing with frames.

The negative graph of h_{13} is A_{31} (this subspace is the negative graph of h_{31}, and hence also of the inverse h_{13}). So it is sufficient to show that the negative graph of $h_{23} \circ h_{12}$ is A_{31}. But

$$\begin{aligned} \mathrm{graph}(-h_{23} \circ h_{12}) &= \{x - z \mid x \in A_1, z \in A_3, z = (h_{23} \circ h_{12})(x)\} \\ &= \{x - h_{12}(x) + h_{12}(x) - h_{23}(h_{12}(x)) \mid x \in A_1, z \in A_3\} \\ &\subseteq (A_1 + A_3) \cap (A_{12} + A_{23}) = A_{31}, \end{aligned}$$

and the last inclusion is, for dimension reasons, actually an equality.

(2) Suppose we are given a frame $\bar{A}$ in some vector space W. Set $U := A_2$ and let $v_1, ..., v_n$ be a basis of A_2. A basis of A_1 is defined via $u_i := h_{21}(v_i)$, $i = 1, ..., n$, and a basis of A_3 is defined by $w_i := h_{23}(v_i)$. Since (271) is equivalent to

$$h_{31} \circ h_{23} \circ h_{12} = \mathrm{id}_{A_1}, \tag{272}$$

it follows that $u_i = h_{31}(w_i)$ for each i. Now the linear isomorphism which sends the basis $u_1, ..., u_n, v_1, .., v_n, w_1, .., w_n$ to the basis of $V := U \times U \times U$ built canonically from $v_1, ..., v_n$ has, as its image, a frame of the desired form.

□

REMARK 8.4.6. One might think of (272) as a kind of cocycle condition which says that that the endomorphism obtained from the "loop" $A_1 \xrightarrow{h_{12}} A_2 \xrightarrow{h_{23}} A_3 \xrightarrow{h_{31}} A_1$ is trivial.

Given a frame $\bar{A}$ on V, with $\dim V = 3k$, we define a **frame basis** for $\bar{A}$ to be an ordered basis

$$\{u_1, ..., u_k, v_1, ..., v_k, w_1, ..., w_k\}$$

of V such that

- $\{u_1, ..., u_k\}$ is a basis of A_1, $\{v_1, ..., v_k\}$ is a basis of A_2, $\{w_1, ..., w_k\}$ is a basis of A_3,
- $h_{12}(u_i) = v_i, \;\; h_{23}(v_i) = w_i, \;\; h_{31}(w_i) = u_i$ for all $i = 1, ..., k$.

A frame basis always exists (c.f. the proof of Lemma 8.4.5).

We define an **augmented frame** in V to be a frame $\bar{A}$ in V together with a subspace $C \subseteq V$ such that $A_1 + A_2 = A_1 \oplus C$. This latter condition says that C is the negative graph of a linear map $h : A_2 \to A_1$ (which is uniquely determined by C). In particular, an augmented frame determines uniquely an endomorphism

$$\eta := h_{12} \circ h : A_2 \longrightarrow A_2$$

which we call the **underlying endomorphism** of the augmented frame. If η is the underlying endomorphism of an augmented frame we write this as $(\bar{A}, \eta)$. As was the case for frames, augmented frames can be viewed as special kinds of poset representations, with the inherited notions of morphism and isomorphism.

LEMMA 8.4.7. *Let $(\bar{A}, \eta)$ in V and $(\bar{A}', \eta')$ in V' be augmented frames. There is a bijective correspondence between morphisms $(A_2, \eta) \to (A'_2, \eta')$ and morphisms $(\bar{A}, \eta) \to (\bar{A}', \eta')$.*

When $(A_2, \eta) = (A'_2, \eta')$, this gives an isomorphism between the endomorphism algebras of (A_2, η) and $(\bar{A}, \eta)$.

PROOF. Suppose $f : V \to V'$ is a morphism of augmented frames. In particular then $f(A_1) \subseteq A'_1$, $f(A_2) \subseteq A'_2$, $f(A_{12}) \subseteq A'_{12}$, and $f(C) \subseteq C'$, which implies that the diagrams

$$\begin{array}{ccc} A_2 & \xrightarrow{f_{|A_2}} & A'_2 \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ A_1 & \xrightarrow{f_{|A_1}} & A'_1 \end{array} \qquad\qquad \begin{array}{ccc} A_1 & \xrightarrow{f_{|A_1}} & A'_1 \\ {\scriptstyle h_{12}}\downarrow & & \downarrow{\scriptstyle h'_{12}} \\ A_2 & \xrightarrow{f_{|A_2}} & A'_2 \end{array}$$

commute. Stacking the second diagram under the first, we obtain that

$$\begin{array}{ccc} A_2 & \xrightarrow{f_{|A_2}} & A'_2 \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \eta'} \\ A_2 & \xrightarrow{f_{|A_2}} & A'_2 \end{array}$$

commutes, as desired.

Conversely, if we have a linear map $f : A_2 \to A'_2$ such that $f \circ \eta = \eta' \circ f$, then it extends to a morphism $\hat{f}$ of augmented frames by setting $\hat{f} = h'_{21} \circ f \circ h_{12}$ on A_1 and $\hat{f} = h'_{23} \circ f \circ h_{32}$ on A_3.

It is easy to see that the thus defined operations "restriction from V to A_2" and "extension from A_2 to V" are mutually inverse.

Now assume $(A_2, \eta) = (A'_2, \eta')$. To show that the mutually inverse "extension" and "restriction" maps define algebra isomorphisms between the respective endomorphism algebras of (A_2, η) and $(\bar{A}, \eta)$, it is sufficient to check that one of these maps is a morphism of algebras. This is easiest to check for the restriction map: the operation of restriction is clearly compatible with composition, addition, scalar multiplication, and the units in the respective endomorphism algebras. □

8.4.3. Framed sextuples. Given an augmented frame $(\bar{A}, C) = (\bar{A}, \eta)$ in V we define an associated sextuple $\mathcal{S}_\eta$ in V by

$$\begin{array}{ll} I_1 = A_1 & C_1 = A_1 + A_2 \\ I_2 = A_3 & C_2 = A_2 + A_3 \\ I_3 = (C + A_3) \cap (A_1 + A_{23}) & C_3 = A_{12} + I_3 \end{array} \tag{273}$$

A sextuple which is isomorphic to one of this type will be called a **framed sextuple**.

LEMMA 8.4.8. *Let $(\bar{A}, \eta)$ and $(\bar{A}', \eta')$ be augmented frames in V and V', respectively .*

(1) *Given a sextuple $\mathcal{S}_\eta$, the underlying augmented frame can be recovered via*

$$\begin{array}{ll} A_1 = I_1 & A_{12} = C_1 \cap C_3 \\ A_2 = C_1 \cap C_2 & A_{23} = (I_3 + I_1) \cap C_2 \\ A_3 = I_2 & A_{31} = (A_1 + A_3) \cap (A_{12} + A_{23}) \\ C = (I_2 + I_3) \cap C_1 & \end{array} \tag{274}$$

(2) *If $\mathcal{S}$ is a sextuple such that the expressions (274) define an augmented frame, then $\mathcal{S}$ is a framed sextuple, i.e. of the form (273).*
(3) *A linear map $f : V \to V'$ is a morphism $(\bar{A}, \eta) \longrightarrow (\bar{A}', \eta')$ if and only if it is a morphism $\mathcal{S}_\eta \longrightarrow \mathcal{S}_{\eta'}$.*

(4) *Any sextuple $\mathcal{S}_\eta$ built from an augmented frame $(\bar{A}, \eta)$ is isomorphic to one with the following form, where $U = A_2$:*

$$\begin{array}{ll} V = U \times U \times U & \\ I_1 = U \times 0 \times 0 & C_1 = U \times U \times 0 \\ I_2 = 0 \times 0 \times U & C_2 = 0 \times U \times U \\ I_3 = \{(-\eta x, x, -x) \mid x \in U\} & C_3 = \{(x, -x, 0) \mid x \in U\} + I_3. \end{array} \tag{275}$$

Proof. (1) This can be checked using elementary linear algebra resp. modular lattice calculations. For example, $C_1 \cap C_2 = (A_1 + A_2) \cap (A_2 + A_3) = A_2$, since by assumption $V = A_1 \oplus A_2 \oplus A_3$. To see that $C = (I_2 + I_3) \cap C_1$, we plug in the definitions of I_2, I_3 and C_1 and calculate

$$\begin{aligned} (I_2 + I_3) \cap C_1 &= (A_3 + [(C + A_3) \cap (A_1 + A_{23})]) \cap (A_1 + A_2) \\ &\subseteq [(C + A_3) \cap (A_1 + A_{23} + A_3)] \cap (A_1 + A_2) \\ &= [(C + A_3) \cap V] \cap (A_1 + A_2) = C. \end{aligned}$$

The obtained inclusion is actually an equality, since

$$\dim(I_3 + I_2) \cap C_1 = \dim(I_1 \oplus I_3) + \dim C_1 - \dim(I_2 + I_3 + C_1) = 1/3 \dim V = \dim C.$$

The equations for A_{12} and A_{23} can be checked in a similar manner, and the equation for A_{31} holds by the definition of a frame.

(2) Assuming that (274) defines an augmented frame, we need to show that the relations (273) hold.

The expressions for I_1 and I_2 are trivially satisfied, so it remains to show the expressions for I_3, C_1, C_2, and C_3.

We first make some intermediate observations:

$$A_1 + C = A_1 + C_1 \cap (A_3 + I_3) = C_1 \cap (A_1 + A_3 + I_3), \tag{276}$$

using the modular law for the last equality; from (276) and the assumption that we have an augmented frame,

$$\begin{aligned} V = A_1 + A_2 + A_3 = A_1 + C + A_3 &= C_1 \cap (A_1 + A_3 + I_3) + A_3 \\ &= (C_1 + A_3) \cap (A_1 + A_3 + I_3), \end{aligned} \tag{277}$$

again via the modular law for the last equality. But (277) implies that

$$V = C_1 + A_3 = A_1 + A_3 + I_3, \tag{278}$$

and using (278) we find that

$$A_2 + A_3 = C_1 \cap C_2 + A_3 = C_2 \cap (C_1 + A_3) = C_2, \tag{279}$$

where the application of modular law is justified since $A_3 = I_2 \subseteq C_2$ by assumption. This is the desired expression for C_2.

The expression for C_1 now follows from (278) and

$$A_1 + A_2 = A_1 + C = A_1 + C_1 \cap (I_2 + I_3) = C_1 \cap (A_1 + I_2 + I_3) = C_1, \tag{280}$$

using modularity via the fact that $A_1 = I_1 \subseteq C_1$.

To obtain the expression for I_3, we first note that

$$C + A_3 = (I_2 + I_3) \cap C_1 + A_3 = (I_2 + I_3) \cap (C_1 + A_3) = I_2 + I_3 \tag{281}$$

since $A_3 = I_2 \subseteq I_2 + I_3$ and, via (278), $C_1 + A_2 = V$. Now,

$$\begin{aligned}(C + A_3) \cap (A_1 + A_{23}) &= (C + A_3) \cap (A_1 + (I_3 + A_1) \cap C_2)\\ &\overset{(279)}{=} (C + A_3) \cap (A_1 + (I_3 + A_1) \cap (A_2 + A_3))\\ &\overset{\text{mod}}{=} (C + A_3) \cap (I_3 + A_1) \cap (A_1 + A_2 + A_3)\\ &\overset{(278)}{=} (C + A_3) \cap (A_1 + I_3) \overset{(281)}{=} I_3 + (C + A_3) \cap A_1\\ &= I_3 + (C \cap A_1) = I_3,\end{aligned}$$

using for the second-to-last equality that $C \subseteq A_1 \oplus A_2$ and $V = A_1 \oplus A_2 \oplus A_3$, and for the last equality that $C \cap A_1 = 0$ by definition of an augmented frame.

Finally, the desired expression for C_3 follows from

$$\begin{aligned}A_{12} + I_3 = (C_1 \cap C_3) + I_3 &\overset{\text{mod}}{=} C_3 \cap (C_1 + I_3)\\ &= C_3 \cap (A_1 + A_2 + (C + A_3) \cap (A_1 + A_{23}))\\ &\overset{\text{mod}}{=} C_3 \cap (A_2 + (A_1 + C + A_3) \cap (A_1 + A_{23}))\\ &= C_3 \cap (A_2 + A_1 + A_{23}) = C_3,\end{aligned}$$

using in the second line the already-obtained expressions for C_1 and I_3, and in the last line the fact that $A_1 + C + A_3 = A_1 + A_2 + A_3 = V$.

(3) Suppose first that f is a morphism of augmented frames. Since the subspaces I_i,C_i and I'_i, C'_i of the respective associated sextuples can be expressed entirely via lattice terms built only of subspaces in the respective augmented frames, it follows that $f(I_i) \subseteq I'_i$ and $f(C_i) \subseteq C'_i$ for each i.

Similarly, by part 1 above the underlying augmented frames can be expressed via lattice terms built from subspaces in the respective sextuples, so f is a morphism of augmented frames if it is a morphism of the associated sextuples.

(4) From Lemma 8.4.5, it is clear that any augmented frame $(\bar{A}, \eta)$ is isomorphic to one of the following form:

$$\begin{array}{ll} V = U \times U \times U & \\ A_1 = U \times 0 \times 0 & A_{12} = \{(x, -x, 0) \mid x \in U\} \\ A_2 = 0 \times U \times 0 & A_{23} = \{(0, x, -x) \mid x \in U\} \\ A_3 = 0 \times 0 \times U & A_{31} = \{(x, 0, -x) \mid x \in U\} \\ C = \{(-\eta x, x, 0) \mid x \in U\} & \end{array} \tag{282}$$

The sextuple associated to this augmented frame is precisely (275), and by part 3 of this lemma, it is isomorphic to S_η.

□

Remark 8.4.9. Together, parts 1 and 2 give a complete characterization of framed sextuples in lattice theoretic terms: a sextuple is a framed sextuple if and only if the expressions (274) define an augmented frame, and augemented frames are themselves defined in lattice-theoretic terms.

Proposition 8.4.10. *Let (U, η), (U', η') be spaces with endomorphisms.*

(1) *There is a bijective correspondence between morphisms $(U, \eta) \to (U', \eta')$ and morphisms $S_\eta \to S'_\eta$. When $(U, \eta) = (U', \eta')$, this correspondence is an isomorphism of the respective endomorphism algebras. In particular, (U, η) is indecomposable if and only if S_η is.*

(2) *Let* $u_1, ..., u_n, v_1, ..., v_n, w_1, .., w_n$ *and* $u'_1, ..., u'_n, v'_1, ..., v'_n, w'_1, .., w'_n$ *be frame bases of the underlying frames of sextuples* S_η *and* $S_{\eta'}$. *Let* $\hat{f} : S_\eta \to S_{\eta'}$ *be a morphism and* f *its restriction* $f : A_2 = \langle v_1, ..., v_n \rangle \to A'_2 = \langle v'_1, ..., v'_n \rangle$. *If the coordinate matrix of* f *with respect to the bases* $\langle v_1, ..., v_n \rangle$ *and* $\langle v'_1, ..., v'_n \rangle$ *is* M, *then the coordinate matrix of* $\hat{f}$ *with respect to the respective frame bases is*

$$\begin{bmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{bmatrix}. \tag{283}$$

PROOF. (1) The said correspondence is the one defined in the proof of Lemma 8.4.7. By Lemma 8.4.7 and Lemma 8.4.8, 2., it maps morphisms $(U, \eta) \to (U', \eta')$ to morphisms $S_\eta \to S'_\eta$, and when $(U, \eta) = (U', \eta')$, this correspondence is an isomorphism of algebras. The statement about indecomposability follows then from the fact that (U, η) and S_η are each indecomposable if and only if their respective endomorphism algebras are local, and the fact that "local-ness" is preserved under isomorphism.

(2) By definition, $\hat{f} = (h'_{21} \circ f \circ h_{12}) \oplus f \oplus (h'_{23} \circ f \circ h_{32}) : A_1 \oplus A_2 \oplus A_3 \to A'_1 \oplus A'_2 \oplus A'_3$ (c.f. Lemma 8.4.8). The form (283) follows now from the fact that, with respect to frame bases, all of the maps h_{12}, h_{23}, h'_{21}, h'_{32} have coordinate matrices which are the identity matrix.

□

REMARK 8.4.11. The correspondence in Proposition 8.4.10, is functorial.

8.4.4. Identifying framed sextuples. In this section we identify which continuous-type indecomposable sextuples are isomorphic to a framed sextuple.

From Lemma 8.4.1 and Corollary 8.4.2 we see that, for η indecomposable, $S_\eta \simeq \Delta_3(k; 0)$ when η is nilpotent, and $S_\eta \simeq \Delta_2(k; \infty)$ when η has eigenvalue 1. Indeed, if η is nilpotent, then

$$I_3 \cap C_2 = \{(-\eta x, x, -x) \mid x \in U\} \cap (0 \times U \times U) \supseteq \{(0, x, -x) \mid x \in \ker \eta\} \neq 0.$$

And if η has eigenvalue 1 with associated eigenspace U_1, then

$$I_2 \cap C_3 = (0 \times 0 \times U) \cap \{(y - \eta x, x - y, -x) \mid x, y \in U\} \supseteq \{(0, 0, -x) \mid x \in U_1\} \neq 0.$$

We will call an indecomposable endomorphism **exceptional** if it is nilpotent or has 1 as eigenvalue. Note that any non-exceptional indecomposable endomorphism is an automorphism.

It remains now to identify the sextuples S_η, with η non-exceptional, in terms of the classification discussed in the previous section.

PROPOSITION 8.4.12. *On the level of isomorphism classes (and for fixed ambient dimension* $3k$*), there is a one-to-one correspondence between the sextuples* S_η, *and the sextuples* $\Delta(k; \gamma)$, *where* γ *and* η *are non-exceptional indecomposable endomorphisms. If one views both* η *and* γ *as endomorphisms of* $\mathbf{k}^k$, *the correspondence is given by*

$$\eta = \frac{\gamma}{\gamma - 1} \tag{284}$$

or, in inversely, $\gamma = \frac{\eta}{\eta - 1}$.

PROOF. We start by considering an indecomposable continuous-type sextuple $\Delta(k;\gamma)$, with γ non-exceptional. From Section 8.4.1, we know that $\Delta(k;\gamma)$ is isomorphic to a sextuple which has the form

$$V = X_1 \oplus X_2 \oplus X_3, \quad C_i = X_i + X_{i+1}, \quad I_i = \mathrm{Im}\,(\alpha_i \oplus \beta_i) = \Gamma(\beta_i \alpha_i^{-1})$$

where $\alpha_i : Y_i \to X_i$ and $\beta : Y_i \to X_{i+1}$ are all invertible, and $\gamma = \beta_1 \circ \alpha_1^{-1}$. In particular, we can identify the Y_i with the I_i (since the maps $\alpha_i \oplus \beta_i$ are injective), and we have

$$C_i = X_i \oplus Y_i = X_{i+1} \oplus Y_i \quad i = 1, 2, 3. \tag{285}$$

To show that $\Delta(k;\gamma) \simeq S_\eta$ for some endomorphism η, we "guess" an underlying augmented frame (using Lemma 8.4.8 to make our ansatz), and we show that this is an augmented frame whose associated sextuple is isomorphic to $\Delta(k;\gamma)$. In this case we know that η must be non-exceptional, since $S_\eta \simeq \Delta_3(k;0)$ or $S_\eta \simeq \Delta_2(k;\infty)$ when η is exceptional.

As our ansatz for the underlying augmented frame associated to $(V;C_i,I_i)$, we set

$$\begin{aligned}
&A_1 = Y_1 && A_{12} = (X_1 + X_2) \cap (X_3 + X_1) = X_1\\
&A_2 = (X_1 + X_2) \cap (X_2 + X_3) = X_2 && A_{23} = (Y_3 + Y_1) \cap (X_2 + X_3)\\
&A_3 = Y_2 && A_{31} = (A_1 + A_3) \cap (A_{12} + A_{23})\\
&C = (Y_2 + Y_3) \cap (X_1 + X_2) &&
\end{aligned}$$

and check that this defines an augmented frame. In order to verify a $\oplus$-relation, it suffices to check the $+$- or the $\cap$-relation, if the dimensions of the subspaces involved are known and add up properly. Thus, to see that $V = A_1 \oplus A_2 \oplus A_3$, it suffices to note that

$$A_1 + A_2 + A_3 = Y_1 + X_2 + Y_2 = X_1 + X_2 + Y_2 = X_1 + X_2 + X_3 = V.$$

Similarly, $A_{12} \oplus A_1 = A_{12} \oplus A_2 = A_1 + A_2$ follows from

$$\begin{aligned}
A_{12} + A_1 &= X_1 + Y_1 = X_1 + X_2 = A_1 + A_2\\
A_{12} + A_2 &= X_1 + X_2 = A_1 + A_2.
\end{aligned}$$

To see $A_{23} \oplus A_2 = A_{23} \oplus A_3 = A_2 + A_3$, note that

$$\begin{aligned}
A_{23} + A_2 &= (Y_1 + Y_3) \cap (X_2 + X_3) + X_2 = (Y_1 + Y_3 + X_2) \cap (X_2 + X_3)\\
&= X_2 + X_3 = X_2 + Y_2 = A_2 + A_3\\
A_{23} + A_3 &= (Y_1 + Y_3) \cap (X_2 + X_3) + Y_2 = (Y_1 + Y_3 + Y_2) \cap (X_2 + X_3)\\
&= X_2 + X_3 = X_2 + Y_2 = A_2 + A_3
\end{aligned}$$

using modularity (and that $X_2, Y_2 \subseteq X_2 + X_3$) to obtain the second equality in each line, respectively. At this point it is not yet clear that the lefthand sums are direct, since the dimension of A_{23} is not yet determined. This can be checked directly, for example:

$Y_1 + Y_3 + X_2 = X_1 + Y_3 + X_2 = X_1 + X_3 + X_2 = V$ whence $A_{23} \cap A_2 = (Y_1 + Y_3) \cap X_2 = 0$

and

$$A_{23} \cap A_3 = (Y_1 + Y_3) \cap Y_2 = 0.$$

Finally, $A_1 \oplus C = A_1 + A_2$ because

$$\begin{aligned}
&A_1 + C = Y_1 + (Y_2 + Y_3) \cap (X_1 + X_2) = (Y_1 + Y_2 + Y_3) \cap (X_1 + X_2) = X_1 + X_2 = A_1 + A_2\\
&A_1 \cap C = Y_1 \cap (Y_2 + Y_3) = 0
\end{aligned}$$

using modularity in the first line (with the fact that $Y_1 \subseteq X_1 + X_2$).

This establishes that we have an augmented frame. It remains to verify that the original sextuple (V, I_i, C_i) is the one associated to this augmented frame, i.e. we check that the equations (273) hold:

- $I_1 = Y_1 = A_1$
- $I_2 = Y_2 = A_3$
- $I_3 = Y_3 = (Y_2 + Y_3) \cap (Y_1 + Y_3) = (C + A_3) \cap (A_1 + A_{23})$,
  using for the last equality that
  $$\begin{aligned} Y_2 + Y_3 &= (Y_2 + Y_3) \cap (X_1 + X_2 + X_3) = (Y_2 + Y_3) \cap (X_1 + X_2 + Y_2) \\ &= (Y_2 + Y_3) \cap (X_1 + X_2) + Y_2 = C + A_3 \end{aligned}$$
  and
  $$\begin{aligned} Y_1 + Y_3 &= (Y_1 + Y_3) \cap (X_1 + X_2 + X_3) = (Y_1 + Y_3) \cap (Y_1 + X_2 + X_3) \\ &= Y_1 + (Y_1 + Y_3) \cap (X_2 + X_3) = (A_1 + A_{23}) \end{aligned}$$
- $C_1 = A_1 + A_2 = Y_1 + X_2 = X_1 + X_2 = C_1$
- $C_2 = A_2 + A_3 = X_2 + Y_2 = X_2 + X_3 = C_2$
- $C_3 = A_{12} + (C + A_3) \cap (A_1 + A_{23}) = A_{12} + I_3 = X_1 + Y_3 = X_1 + X_3 = C_3$

This establishes that every sextuple $\Delta(k; \gamma)$ is isomorphic to some S_η, with η non-exceptional. It remains now to show that every S_η with η non-exceptional is isomorphic to some $\Delta(k; \gamma)$. For this it is sufficient to prove the formula (284) and note that this formula defines a bijection (in fact an involution) of the set of non-exceptional indecomposable endomorphisms of $\mathbf{k}^k$.

Consider again a sextuple of type $\Delta(k; \gamma)$. Set $X = \mathbf{k}^k$. We'll use normal forms which are isomorphic to the ones (261): let $V = X \times X \times X$, and

$$X_1 = X \times 0 \times 0, \ X_2 = 0 \times X \times 0, \ X_3 = 0 \times 0 \times X.$$

Thus normal forms for these sextuple are

$$\begin{array}{ll} I_1 = \Gamma(\beta_1 \alpha_1^{-1}) = \{(x, -\gamma x, 0) \mid x \in X\} & C_1 = X \times X \times 0 \\ I_2 = \Gamma(\beta_2 \alpha_2^{-1}) = \{(0, x, x) \mid x \in X\} & C_2 = 0 \times X \times X \\ I_3 = \Gamma(\beta_3 \alpha_3^{-1}) = \{(x, 0, x) \mid x \in X\} & C_3 = X \times 0 \times X. \end{array} \tag{286}$$

Set $g_i := \beta_i \alpha_i^{-1}$ and $g := g_1 g_3 g_2$. Note that $g = -\gamma$ when these are viewed as maps $X \to X$.

For such a sextuple, we compute the endomorphism η underlying the associated augmented frame. By definition, $\eta = h_{12} \circ h$, so we need to compute h_{12} and h. From the first part of this proof we know that for this frame

$$\begin{aligned} A_1 &= I_1 \\ A_2 &= X_2 \\ A_3 &= I_2 \\ A_{12} &= \Gamma(-h_{12}) = X_1 \\ C &= \Gamma(-h) = (I_2 + I_3) \cap (X_1 + X_2). \end{aligned}$$

In particular one finds easily that

$$h_{12} : A_1 \to A_2, \ (x, g_1 x, 0) \mapsto (0, g_1 x, 0),$$

$C = \{(-g_3 g_2 x, x, 0) \mid x \in X\}$, and

$$h : A_2 \to A_1, \ (0, x, 0) \mapsto (g_3 g_2 (g + 1)^{-1} x, g(g + 1)^{-1} x, 0).$$

It follows that

$$\eta : A_2 \to A_2,\ (0,x,0) \mapsto (0, g(g+1)^{-1}x, 0).$$

Thus, viewed as endomorphisms of X, $\eta = \frac{\gamma}{1-\gamma}$. □

REMARK 8.4.13. Since η and γ commute, if γ has an eigenvalue, say λ_γ, then η will also have an eigenvalue, say λ_η, and any eigenvector for λ_γ will also be an eigenvector for λ_η. In this case, $\lambda_\eta = \lambda_\gamma/(\lambda_\gamma - 1)$ and $\lambda_\gamma = \lambda_\eta/(\lambda_\eta - 1)$.

8.4.5. Duals of continuous sextuples. In this section, we identify the duals of indecomposable continuous-type sextuples. We recall:

- The **dual** of a sextuple $(V; C_i, I_i)$ is $(V^*; I_i^\circ, C_i^\circ)$, where, for $U \subseteq V$, the subspace $U^\circ = \{f \in V^* \mid f(U) = 0\}$ is the annihilator of U.
- A sextuple is **self-dual** if it admits an isomorphism (of poset representations) to its dual. A pair of sextuples is called **mutually dual** if each is isomorphic to the dual of the other.
- The operation of taking the annihilator obeys the rules

$$(U_1 + U_2)^\circ = U_1^\circ \cap U_2^\circ \quad \text{and} \quad (U_1 \cap U_2)^\circ = U_1^\circ + U_2^\circ,$$

  for any subspaces U_1, U_2.

From the structure of the dimension vector of indecomposable continuous-type sextuples it follows that the dual of a continuous-type sextuple is again of continuous type. Identifying the duals of the exceptional continuous-type sextuples is easiest.

LEMMA 8.4.14. *The indecomposable sextuples of type $\Delta_1(k;1)$ and $\Delta_2(k;1)$ are mutually dual.*

PROOF. This follows from Corollary 8.4.2: if $(V; C_i, I_i)$ is a sextuple of type $\Delta_1(k;1)$, then $\dim(I_1 + I_2 + I_3) = \dim V - 1$. Thus for the dual $(V'; C_i', I_i')$ will hold $C_1' \cap C_2' \cap C_3' = I_1^\circ \cap I_2^\circ \cap I_3^\circ = (I_1 + I_2 + I_3)^\circ = 1$. This implies, by Corollary 8.4.2 that the dual is of type $\Delta_2(k;1)$ (the dual must be indecomposable and of continuous type). □

LEMMA 8.4.15. *The indecomposable sextuples $\Delta_i(k;0)$ and $\Delta_{i-1}(k;\infty)$ are mutually dual. In other words, in each ambient dimension $3k$, we have the following three pairs of mutually dual indecomposable sextuples:*

$$\Delta_1(k;0) \text{ and } \Delta_3(k;\infty), \quad \Delta_2(k;0) \text{ and } \Delta_1(k;\infty), \quad \Delta_3(k;0) \text{ and } \Delta_2(k;\infty).$$

PROOF. Suppose $(V; C_i, I_i)$ is a sextuple of type $\Delta_i(k;0)$. By Corollary 8.4.2, this sextuple will satisfy $\dim(I_i \cap C_{i-1}) = 1$. In particular then

$$\dim(I_i + C_{i-1}) = \dim I_i + \dim C_{i-1} - \dim(I_i \cap C_{i-1}) = k + (2k) - 1 = \dim V - 1.$$

The dual sextuple $(V'; C_i', I_i')$ will therefore satisfy

$$\dim(I_{i-1}' \cap C_i') = \dim(C_{i-1}^\circ \cap I_i^\circ) = \dim(I_i + C_{i-1})^\circ = 1.$$

This implies, via Corollary 8.4.2, that $(V'; C_i', I_i')$ is of type $\Delta_{i-1}(k;\infty)$ □

REMARK 8.4.16. From Section 8.4.4 we know that $\Delta_3(k;0)$ and $\Delta_2(k;\infty)$ are isomorphic, respectively, to the indecomposable framed sextuples S_η where η is either nilpotent (in the first case) or has eigenvalue 1 (in the second case). Thus the above shows that the sole two exceptional framed sextuples are dual to one another.

Now we consider the non-exceptional indecomposable continuous-type sextuples $\Delta(k;\gamma)$. From Corollary 8.4.2 it follows that the dual of such a sextuple is again a non-exceptional indecomposable continuous-type sextuple; let $\Delta(k;\gamma')$ be the type of the dual. Our goal now is to determine the relationship between γ and γ'.

From Proposition 8.4.12 we know that $\Delta(k;\gamma)$ and $\Delta(k;\gamma')$ are isomorphic, respectively, to sextuples S_η and $S_{\eta'}$, with η and η' non-exceptional.

PROPOSITION 8.4.17. *Consider a non-exceptional indecomposable endomorphism $\eta \in End(U)$ with associated sextuple $\mathcal{S}_\eta$. Let $\beta := (u_1, ..., u_k, v_1, ..., v_k, w_1, .., w_k)$ be a frame basis for this sextuple. Then:*

(1) *The dual of $\mathcal{S}_\eta$ is isomorphic to $\mathcal{S}_{\eta'}$, with $\eta' = (Id - \eta)^* \in End(U^*)$. Moreover,*
$$(-w_1^*, ..., -w_k^*, v_1^*, ..., v_k^*, -u_1^*, ..., -u_k^*)$$
is a frame basis for $\mathcal{S}_{\eta'}$, where $\beta^* := (u_1^*, ..., w_k^*)$ is the dual basis of β.

(2) *A bijective correspondence*
$$B : \mathcal{S}_\eta \xrightarrow{\sim} \mathcal{S}_{\eta'} \qquad \longleftrightarrow \qquad b : (U, \eta) \xrightarrow{\sim} (U^*, Id - \eta^*)$$
is given by restricting maps B to U.

With respect to the bases β and β^*, any isomorphism $B : \mathcal{S}_\eta \to \mathcal{S}_{\eta'}$ has coordinate matrix of the form
$$H_M := \begin{pmatrix} O & O & -M \\ O & M & O \\ -M & O & O \end{pmatrix},$$
where $M \in \mathbf{k}^{k\times k}$ is the coordinate matrix of the corresponding isomorphism $b : (U, \eta) \to (U^*, \eta')$ with respect to the respective bases $(v_1, ..., v_k)$ and $(v_1^*, ..., v_k^*)$ of U and U^*.

Note, in particular, that B is (skew)-symmetric if and only if the corresponding map b is.

PROOF. By definition, $\mathcal{S}_\eta$ has ambient space $V = U \times U \times U$, and the associated augmented frame $(\bar{A}, \eta)$ is (282); in particular $A_1 = U \times 0 \times 0$, $A_2 = 0 \times U \times 0$ and $A_3 = 0 \times 0 \times U$. We view the dual sextuple $\mathcal{S}_\eta^* = S_{\eta'} = (V'; C_i', I_i')$ as having ambient space $V' = U^* \times U^* \times U^*$, paired with $V = U \times U \times U$ via $(\xi_1, \xi_1, \xi_3) : (u_1, u_2, u_3) \mapsto \xi_1(u_1) + \xi_2(u_2) + \xi_3(u_3)$. The sextuple $\mathcal{S}_{\eta'}$, expressed in terms of the underlying augmented frame of $\mathcal{S}_\eta$, is

$$\begin{aligned} &I_1' = C_1^\circ = A_1^\circ \cap A_2^\circ = 0 \times 0 \times U^* && C_1' = I_1^\circ = A_1^\circ = 0 \times U^* \times U^* \\ &I_2' = C_2^\circ = A_2^\circ \cap A_3^\circ = U^* \times 0 \times 0 && C_2' = I_2^\circ = A_3^\circ = U^* \times U^* \times 0 \\ &I_3' = C_3^\circ = A_{12}^\circ \cap [(C^\circ \cap A_3^\circ) + (A_1^\circ \cap A_{23}^\circ)] && C_3' = I_3^\circ = (C^\circ \cap A_3^\circ) + (A_1^\circ \cap A_{23}^\circ) \end{aligned} \tag{287}$$

We compute the underlying augmented frame $(\bar{A}', \eta')$ of this sextuple, using (282) as an aid for the calculations:

$$\begin{aligned} &A_1' = I_1' = 0 \times 0 \times U^* \\ &A_2' = C_1' \cap C_2' = 0 \times U^* \times 0 \\ &A_3' = I_2' = U^* \times 0 \times 0 \\ &A_{12}' = C_1' \cap C_3' = (I_1 + I_3)^\circ = \{(y, x, -x) \mid x, y \in U\}^\circ = \{(0, \xi, \xi) \mid \xi \in U^*\} \\ &A_{23}' = (I_3' + I_1') \cap C_2' = [(C_3 \cap C_1) + I_2]^\circ = \{(\xi, \xi, 0) \mid \xi \in U^*\} \\ &A_{31}' = (A_1' + A_3') \cap (A_{12}' + A_{23}') \\ &C' = (I_2' + I_2') \cap C_1' = [(C_2 \cap C_3) + I_1]^\circ = \{(0, \xi, (\mathrm{id} - \eta)^*\xi) \mid \xi \in U^*\} \end{aligned} \tag{288}$$

For the computation of C', for example: $C_2 \cap C_3 = \{(y - \eta x, x - y, -x) \mid x, y \in U, y - \eta x = 0\}$, and $I_1 + C_2 \cap C_3 = \{(y, x - \eta x, -x) \mid x, y \in U\}$, so C' is precisely those $(\xi_1, \xi_2, \xi_3) \in U^* \times U^* \times U^*$ such that $\xi_1 = 0$ and $\xi_3 = \xi_2 \circ (\mathrm{id} - \eta)$.

We can now read off the maps $h'_{i,i+1}$ and h' associated to this augmented frame:

- $A'_{12} = \Gamma(-h'_{12})$ implies that $h'_{12} : A'_1 \to A'_2, (0, 0, \xi) \mapsto (0, -\xi, 0)$.
- $A'_{23} = \Gamma(-h'_{23})$ implies that $h'_{23} : A'_2 \to A'_3, (0, \xi, 0) \mapsto (-\xi, 0, 0)$.
- $h'_{31} = (h'_{23} \circ h'_{12})^{-1} : A'_3 \to A'_1, (\xi, 0, 0) \mapsto (0, 0, \xi)$.
- $C' = \Gamma(-h')$ implies that $h' : A'_2 \to A_1, (0, \xi, 0) \mapsto (0, 0, -(\mathrm{id} - \eta)^* \xi)$

In particular, $\eta' = h'_{12} \circ h' = (\mathrm{id} - \eta)^*$. We also see that, taking $(-v_1^*, ..., -v_k^*)$ as a basis of $A'_2 = 0 \times U^* \times 0$, a frame basis is given by $(w_1^*, ..., w_k^*, -v_1^*, ..., -v_k^*, u_1^*, ..., u_k^*)$, since $w_i^* = h'_{21}(-v_i^*)$ and $u_i^* = h'_{23}(-v_i^*)$, $i = 1, .., k$. The remaining statement of the current proposition follows now from Proposition 8.4.10. In particular, any isomorphism $\mathcal{S}_\eta \to \mathcal{S}_{\eta'}$ has, with respect to the bases β and β^*, coordinate matrix of the form H_M, where M is the coordinate matrix of an isomorphism $U \to U^*$ such that $MAM^{-1} = \mathrm{id} - A^t$. □

Corollary 8.4.18. *The sextuples $\Delta(k; \gamma)$ and $\Delta(k; (\gamma^{-1})^*)$ are mutually dual.*

Proof. Viewing γ and η as endomorphisms of the same space, and similarly for γ' and η', we have by (284) that $\eta = \frac{\gamma}{\gamma - 1}$ and $\gamma' = \frac{\eta'}{\eta' - 1}$. Substituting $\eta' = 1 - \eta^*$ in the latter equation, using the former equation and simplifying gives $\gamma' = (\gamma^{-1})^*$. □

Corollary 8.4.19. *A sextuple $\Delta(k; \gamma)$ is self-dual if and only if $(\mathbf{k}^k, \gamma)$ and $((\mathbf{k}^k)^*, (\gamma^{-1})^*)$ are isomorphic endomorphisms (in the sense of Section 7.2). In this case, if γ has an eigenvalue λ_γ, then $\lambda_\gamma = -1$.*

Proof. The first part follows from Corollary 8.4.18 and the fact that, by Propositions 8.4.12 and 8.4.10, $\Delta(k; \gamma_1)$ and $\Delta(k; \gamma_2)$ are isomorphic sextuples if and only if γ_1 and γ_2 are isomorphic endomorphisms. The statement about eigenvalues follows from the fact that γ^* has the same eigenvalue as γ (when such exists), and that γ cannot have eigenvalue 1 (by the assumption that γ is non-exceptional). □

8.5. Continuous non-split isotropic triples over $\mathbb{C}$ and $\mathbb{R}$

We turn now to the classification of non-split isotropic triples of continuous type. The special cases where the ground field $\mathbf{k}$ is either $\mathbb{C}$ or $\mathbb{R}$ are of particular interest. We treat these cases first, which allows for a simpler analysis. Furthermore, the case $\mathbf{k} = \mathbb{R}$ provides basic intuition for understanding the more involved classification over perfect fields, which we undertake in the next section.

Throughout this and the next section, N will always denote a nilpotent square matrix which has "1" everywhere on the first upper off-diagonal and all other entries "0", and whose size will be specified, or clear, depending on the context. We call N the standard indecomposable nilpotent matrix (of a given size). For example, if the size happens to be 3×3, then

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

For block matrices, we will use the following convention, which generalizes multiplication of matrices by scalars: if M is a block matrix with square blocks of size $l \times l$, and if K is an $l \times l$ matrix, then KM will denote the block matrix whose blocks consist of the blocks of M each multiplied on the left by K. We define MK similarly.

8.5.1. Classification in case of eigenvalue in ground field. Given a $3k$-dimensional indecomposable sextuple $\mathcal{S}_\eta$, we call a frame basis

$$(u_1, ..., u_k, v_1, ..., v_k, w_1, ..., w_k)$$

a **Jordan** basis for $\mathcal{S}_\eta$ if $(v_1, ..., v_k)$ is a Jordan basis for η for some (unique) $\lambda \in \mathbf{k}$, i.e. $\eta(v_1) = \lambda v_1$, $\eta(v_j) = v_{j-1} + \lambda v_j$ for $j > 1$. Note that any Jordan basis for η extends to one for $\mathcal{S}_\eta$.

THEOREM 8.5.1. *Consider an indecomposable sextuple $\mathcal{S}_\eta$ of dimension $3k$ with underlying endomorphism η having eigenvalue $\lambda \in \mathbf{k}$; fix a Jordan basis for $\mathcal{S}_\eta$. Let $M \in \mathbf{k}^{k\times k}$ be of the form*

$$\begin{pmatrix} & & \dots & 0 & (-1)^{j+1} \\ & \dots & \dots & \dots & \dots \\ \dots & 0 & 1 & 0 & \dots \\ 0 & -1 & 0 & \dots & \\ 1 & 0 & \dots & & \end{pmatrix}$$

i.e. the entries of M satisfy $m_{ij} = (-1)^{j+1}$ for $i = k - j + 1$, and $m_{ij} = 0$ else.

(1) *The sextuple is isomorphic to its dual if and only if $\lambda = \frac{1}{2}$.*

Now assume $\lambda = \frac{1}{2}$.

(2) *The matrix*

$$H_M = \begin{pmatrix} O & O & -M \\ O & M & O \\ -M & O & O \end{pmatrix}$$

defines a compatible form B which is symplectic if k is even, symmetric if k is odd.

(3) *Up to isomorphism and multiplication with a scalar, B is the only such form. For even k there is no symmetric compatible form, for odd k no symplectic one.*

(4) *For any $0 \neq c \in \mathbf{k}$, cB and B are isometric via an automorphism of the sextuple if and only if c is a square in $\mathbf{k}$.*

(5) *A complete list of compatible symplectic resp. symmetric forms is given by the matrices H_{MQ}, where $Q \in \mathbf{k}^{k\times k}$ is of the form $\sum_{i=0}^{k-1} a_i N^i$, with $a_0 \neq 0$ and $a_i = 0$ for i even.*

REMARK 8.5.2. Observe that the particular value $\lambda = \frac{1}{2}$ arises from the way the endomorphism is related to the sextuple. Other ways would give a different (but also unique) value.

Also note that, setting $\zeta := \eta - \frac{1}{2}$, we have that $\eta = \frac{1}{2} + \zeta$ underlies an indecomposable self-dual framed sextuple if and only if ζ is similar to $-\zeta^*$. Furthermore, η has an eigenvalue if and only if ζ does, and in the case of self-duality, the unique eigenvalue of ζ is 0. The description "$\eta = \frac{1}{2} + \zeta$" is helpful to keep in mind in the proof below, and in the subsequent sections.

Proof of Theorem 8.5.1 With respect to $v_1, \dots, v_k$, the matrix of η is $A = \lambda I + N$. In particular, A is similar to A^t via the permutation matrix P corresponding to reversing the order of the basis.

Proof of 1 and 2. Assume that $\mathcal{S}_\eta$ is isomorphic to its dual. Then by Proposition 8.4.17, $I - A^t$ is similar to A, and hence to A^t. Thus, A is similar to $I - A$. In particular, $I - A$ necessarily also has λ as its unique eigenvalue. On the other hand, given an eigenvector

v for A, we have $(I-A)v = v - \lambda v = (1-\lambda)v$, which means that $1-\lambda$ is an eigenvalue of $I-A$. By indecomposablilty, $I-A$ can only have one eigenvalue, thus $\lambda = 1-\lambda$, i.e. $\lambda = \frac{1}{2}$.

Now, assume $\lambda = \frac{1}{2}$ and define B by H_M. It remains to show that B is compatible, i.e. that H_M defines, with respect to the given basis and its dual, an isomorphism from $\mathcal{S}_\eta$ onto its dual. By Proposition 8.4.17 this means to show that M defines an isomorphism $(A_2, \eta) \to (A_2^*, (\mathrm{id}-\eta)^*)$, i.e. $MAM^{-1} = (I-A)^t$. Here, observe that $M = DP$, where $P = P^{-1}$ has anti-diagonal entries 1, zero else, and where $D = D^{-1}$ is diagonal with entries $d_{jj} = (-1)^{j+1}$. Now, $PAP = A^t = \frac{1}{2}I + N^t$ and $DN^tD = -N^t$, and hence $MAM^{-1} = \frac{1}{2}I + DN^tD = \frac{1}{2}I - N^t = I - A^t$.

Proof of 3 – 5 By Lemma 7.3.3, the algebra E of endomorphisms of A_2 commuting with $A = \frac{1}{2}I + N$ is local and $E = \mathbf{k}\mathrm{id} \oplus \mathrm{Rad}E$; Proposition 8.4.10 establishes an isomorphism of E onto the endomorphism algebra E' of $\mathcal{S}_\eta$ which shows that E' is local, too, with $E' = \mathbf{k}\mathrm{id} \oplus \mathrm{Rad}E'$. Thus, Lemma 7.8.1 applies, proving uniqueness. The proof of 4. can be copied from that of Theorem 8.3.12. Moreover, units in the ring E' are given by block matrices as in (283). Thus, due to 2, any compatible form is given by a matrix H_{MQ} where $Q \in \mathbf{k}^{k\times k}$ is the coordinate matrix of an automorphism of (A_2, η). In particular, this means that Q is of the form $\sum_{l=0}^{k} a_l N^l$ with $a_0 \neq 0$. Thus $MQ = \sum_{l=0}^{k} a_l MN^l$. Note that MN^l is skew-symmetric if $l = k \mod 2$ and symmetric if $l \neq k \mod 2$:

$$(MN^l)_{ij} = \sum_p m_{i,p}(N^l)_{p,j} = m_{i,k-i+1}(N^l)_{k-i+1,j} = (-1)^{k-i} \quad \text{for } i+j = l+k+1$$

$$(MN^l)_{ij} = 0 \quad \text{else,}$$

and so, for $i+j = l+k+1$,

$$(MN^l)^t_{ij}(MN^l)_{ij} = (MN^l)_{ji}(MN^l)_{ij} = (-1)^{k-j}(-1)^{k-i} = (-1)^{k-l-1}.$$

It follows that $MQ = \sum_{i \text{ even}} a_i MN^i + \sum_{i \text{ odd}} a_i MN^i$ is the (unique) decomposition of MQ into its symmetric and skew-symmetric parts: when k is odd, the first summand is the symmetric part and the second summand skew-symmetric; when k is even, the reverse is the case. The second summand is non-invertible, while the first summand is invertible (since $a_0 \neq 0$). Thus all compatible (non-degenerate) forms are of the form H_{MQ}, where $MQ = \sum_{i \text{ even}} a_i MN^i$. For k odd they are symmetric; for k even, they are skew-symmetric. □

Example 8.5.3. Let $k = 4$, and consider the self-dual sextuple $\mathcal{S}_\eta$ with underlying indecomposable endomorphism having eigenvalue $\lambda = \frac{1}{2}$. Then $\mathcal{S}_\eta$ admits only compatible symplectic forms. These are parametrised by the set $\{(a_0, a_2) \in \mathbf{k}^2 \mid a_0 \neq 0\}$ and given, with respect to a fixed Jordan basis and its dual basis, by the matrices

$$H_A = \begin{pmatrix} O & O & -A \\ O & A & O \\ -A & O & O \end{pmatrix},$$

where

$$A = \begin{pmatrix} 0 & 0 & 0 & -a_0 \\ 0 & 0 & a_0 & 0 \\ 0 & -a_0 & 0 & -a_2 \\ a_0 & 0 & a_2 & 0 \end{pmatrix}.$$

8.5.2. Classification over $\mathbb{R}$.

THEOREM 8.5.4. *Let $\mathcal{S}_\eta$ be an indecomposable sextuple over $\mathbb{R}$ of dimension $3k$ with underlying η having no real eigenvalue.*

(1) *The sextuple admits an isomorphism onto its dual if and only if η has, in $\mathbb{C}$, eigenvalues $\lambda = \frac{1}{2} \pm \sqrt{-1}r$, with real $r > 0$ (both of geometric multiplicity 1).*

(2) *Assume the case of self-duality. Then $k = 2l$ and both symmetric and symplectic compatible forms exist. Furthermore:*

   (a) *There is a frame basis (u_i, v_i, w_i) such that the matrix A of η is in real Jordan normal form with respect to the v_i, and there are "canonical" compatible forms B and B', where "$B' = \sqrt{-1}B$". If l is odd, B is symmetric and B' is symplectic; if l is even, B is symplectic and B' is symmetric.*
   With respect to the frame basis, B is given by the matrix $H = H_M$ where $M \in \mathbf{k}^{k\times k}$ has 2×2-blocks M_{ij} with

$$M_{ij} = (-1)^{j+1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ for } i+j = l+1,\ M_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ else}$$

   and the matrix H' of B' is obtained from H by setting $H' = \Im H$, where $\Im$ is the matrix

$$\Im = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

   Multiplication with $\Im$ is understood in sense of "scalar multiplication by '$\sqrt{-1}$' for 2×2 block matrices".

   (b) *All symmetric resp. symplectic compatible forms are given, up to isomorphy and multiplication with ± 1, respectively by*

   H resp. $\Im H$ if l is odd, $\Im H$ resp. H if l is even.

   In particular, multiplying with $\Im$ changes the symmetry of a compatible form. There is no automorphism which is an isometry from H to $-H$ or from $\Im H$ to $-\Im H$.

   (c) *A complete list of compatible symmetric, respectively symplectic, forms is given by the matrices H_{MQ} resp. $\Im H_{MQ}$ where M is as above and where $Q \in \mathbb{R}^{k\times k}$ is of the form $\sum_{i=0}^{l-1} a_i N^{2i}$, $a_i \in \mathbb{R}$, $a_0 \neq 0$, and $a_i = 0$ for i even. These are symmetric, respectively symplectic, if l is odd; symplectic, respectively symmetric, if l is even.*

PROOF. With respect to a suitable basis of A_2, η has coordinate matrix in real Jordan form

$$A = ZI + N^2, \quad \text{where } Z = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

and $a \pm \sqrt{-1}b$ are the complex eigenvalues of η ($b \neq 0$ by hypothesis). Here, ZI is the block diagonal matrix with diagonal blocks Z and N the standard indecomposable nilpotent matrix. In view of Proposition 8.4.17, self-duality means that $(I-A)^t$ is similar to A. Now, $I - A$ and $(I-A)^t$ have the same complex eigenvalues, and these are also those of A, by similarity. Thus, λ is an eigenvalue of A if and only if so is $1-\lambda$ and so both have the same real part which then must be $\frac{1}{2}$. This proves that

$$Z = \begin{pmatrix} \frac{1}{2} & -r \\ r & \frac{1}{2} \end{pmatrix}, r \neq 0$$

On the other hand, any such Z satisfies $(I - Z)^t = Z$, and thus corresponds to a self-dual sextuple.

Now, to prove that H_S is a compatible form, we can mimic the proof of 2 in Theorem 8.5.1, replacing scalar matrix entries there by 2×2-blocks here: 0, 1, and -1 are replaced by zero, unit, and negative unit matrix, $\frac{1}{2}$ by B, and m_{ij} by M_{ij}. Thus H_S defines an isomorphism onto the dual. The further statements are then obvious.

Proof of (2b). Observe that matrices

$$Z = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

belong to a subring F of $\mathbb{R}^{2\times 2}$ which is isomorphic to $\mathbb{C}$. F is the field in $\mathbb{R}^{2\times 2}$ obtained by adjoining, for example, $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ to the field $\{\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^{2\times 2}$. For any $Z \in F$, let ZI denote the block diagonal matrix in $\mathbb{R}^{k\times k}$ with diagonal blocks Z. Now, C is in the (coordinatized) endomorphism algebra E of (A_2, η) if and only if it commutes with the matrix A of η. By Proposition 8.6.11 below, E consists of the matrices C of the form

$$C = \sum_{i=0}^{l-1} Z_i N^{2i}, \quad Z_i \in F.$$

By Proposition 8.4.10, $\mathcal{S}_\eta$ has endomorphism algebra consisting of the block-diagonal matrices CI, with $C \in E$ (i.e. there are three diagonal blocks, each one a copy of a given $C \in E$). In particular, these matrices commute with H_M. Now the proof of Lemma 7.8.1 generalizes[13] to yield uniqueness of compatible forms up to isomorphism and multiplication with $Z \in F$; that is, up to isomorphism, compatible forms are of the form $ZH_M = H_{ZM}$ with

$$Z = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \in F \backslash \{0\}.$$

If $y = 0$, then ZM has blocks

$$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \begin{pmatrix} -x & 0 \\ 0 & -x \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \cdots$$

along its anti-diagonal, and so $ZH_M = xH_M$. In this case, the scaling map $1/\sqrt{|x|} \cdot \mathrm{Id}$ is an isometric isomorphism to from xH_M to $\mathrm{sign}(x)H_M$.

If $x = 0$, we have blocks

$$\begin{pmatrix} 0 & -y \\ y & 0 \end{pmatrix}, \begin{pmatrix} 0 & y \\ -y & 0 \end{pmatrix}, \begin{pmatrix} 0 & -y \\ y & 0 \end{pmatrix} \cdots$$

on the anti-diagonal of M, and so $ZH_M = y\Im H_M$. The scaling map $1/\sqrt{|y|} \cdot \mathrm{Id}$ is an isometric isomorphism to from $y\Im H_M$ to $\mathrm{sign}(y)\Im H_M$.

If $x \neq 0$ and $y \neq 0$ then we have

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}, \begin{pmatrix} -x & y \\ -y & -x \end{pmatrix}, \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \cdots$$

on the diagonal of ZM, and neither symmetry nor skew-symmetry.

It remains to show that there is no isometric isomorphism from H_M to $-H_M$ (such a map would also give an isometric isomorphism between $\Im H$ and H). Assume that f were

[13] For a more thorough discussion, see Subsection 8.6.5.

such an automorphism and C the matrix of $f|_{A_2}$. From the above description of $C \in E$ one reads off that C is upper block triangular with

$$c_{11} = c_{2k-1,2k-1}, \quad c_{21} = c_{2k,2k-1}.$$

On the other hand, by inspection of H_M we have

$$B(v_1, v_{2k-1}) = B(v_2, v_{2k}) = 1, \quad B(v_1, v_i) = 0 \text{ for } i \neq 2k-1, \quad B(v_2, v_i) = 0 \text{ for } i \neq 2k.$$

Thus, one would get the contradiction

$$\begin{aligned} -1 = -B(v_1, v_{2k-1}) &= B(fv_1, fv_{2k-1}) = B(c_{11}v_1 + c_{21}v_2, \sum_{i=1}^{2k} c_{i,2k-1}v_i) \\ &= B(c_{11}v_1, c_{1,2k-1}v_{2k-1}) + B(c_{21}v_2, c_{2k,2k-1}v_{2k}) = c_{11}^2 + c_{21}^2. \end{aligned}$$

Proof of (2c). As in the proof of 4. in Theorem 8.5.1 one obtains matrices as stated above, but initially with $a_i \in F$. To have symmetry or skew-symmetry in MQ, though, all a_i have to be in $\mathbb{R}I$ or $\mathbb{R}\Im I$. Conversely, this grants symmetry resp. skew symmetry, depending on the parity of l. □

EXAMPLE 8.5.5. Let $k = 6$, and let η be an indecomposable endomorphism over $\mathbb{R}$, with complex eigenvalues $\frac{1}{2} \pm \sqrt{-1}r$, $r > 0$. The corresponding sextuple $\mathcal{S}_\eta$ is self-dual (and it lives in ambient dimension $3k = 18$); compatible symmetric resp. symplectic forms are given by matrices H_M, where M is of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_0 \\ 0 & 0 & -a_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_0 & 0 & 0 \\ a_0 & 0 & 0 & 0 & a_1 & 0 \\ 0 & a_0 & 0 & 0 & 0 & a_1 \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -a_0 \\ 0 & 0 & 0 & 0 & a_0 & 0 \\ 0 & 0 & 0 & a_0 & 0 & 0 \\ 0 & 0 & -a_0 & 0 & 0 & 0 \\ 0 & -a_0 & 0 & 0 & 0 & -a_1 \\ a_0 & 0 & 0 & 0 & a_1 & 0 \end{pmatrix}$$

for $(a_0, a_1) \in \mathbb{R}^2$, $a_0 \neq 0$.

8.5.3. Hamiltonian vector fields over $\mathbb{R}$ from non-split framed sextuples. In this section we return to the connection, discussed in Section 8.2.8, between linear hamiltonian vector fields and isotropic triples. We show here that each non-split isotropic triple (over $\mathbb{R}$) has an associated linear hamiltonian vector field.

Let S_η be an indecomposable self-dual continuous-type sextuple over $\mathbb{R}$. We know that the spectrum in $\mathbb{C}$ of η must be $\{\frac{1}{2} \pm \sqrt{-1}r\}$ for some value of $r \geq 0$. Suppose that we make S_η into an isotropic triple by choosing a frame basis and choosing a compatible symplectic form given by a matrix H_{MQ} as in Theorem 8.5.1 or Theorem 8.5.4 (depending on whether η has the eigenvalue $\frac{1}{2}$ or not). Set $T := MQ$ and let A denote the coordinate matrix of η with respect to the frame basis. Recall that when H_T is skew-symmetric, as we have assumed, then so is T. From the normal form given in Lemma 8.4.8, part 4, we can assume that S_η has the form

$$\begin{aligned} &V = U \times U \times U \\ &I_1 = U \times 0 \times 0 && C_1 = U \times U \times 0 \\ &I_2 = 0 \times 0 \times U && C_2 = 0 \times U \times U \\ &I_3 = \{(-\eta x, x, -x) \mid x \in U\} && C_3 = \{(x, -x, 0) \mid x \in U\} + I_3, \end{aligned} \tag{289}$$

and from Proposition 8.4.17 it follows that T is the coordinate matrix of an isomorphism $(U, \eta) \to (U^*, 1 - \eta^*)$. In other words, $TA = (1 - A^t)T$.

We claim now that an isotropic triple whose underlying sextuple is (289) satisfies the hypotheses of Proposition 8.2.7, which gives sufficient conditions for constructing an associated hamiltonian vector field. To verify the hypotheses, note first that clearly $V = I_1 \oplus I_2 \oplus I_3$. Second, we must check that $I_I + I_j$ is a symplectic subspace for all $i \neq j$. For this, we check that

$$(I_i + I_j) \cap (I_i + I_j)^{\perp} = (I_i + I_j) \cap C_i \cap C_j = 0.$$

For $i = 1, j = 2$,

$$(I_1 + I_2) \cap C_1 \cap C_2 = (U \times 0 \times U) \cap (0 \times U \times 0) = 0.$$

For $i = 3, j = 1$,

$$(I_3 + I_1) \cap C_3 \cap C_1 = \{(-\eta x + y, x, -x) \mid x, y \in U\} \cap \{(z, -z, 0) \mid y \in U\} = 0,$$

since an element $(-\eta x + y, x, -x)$ of this intersection would need to satisfy $-x = 0$, and hence $-\eta x = 0$. But then $(y, 0, 0) = (z, -z, 0)$ holds for some $z \in U$ only if $y = 0$.

Finally, for $i = 2, j = 3$,

$$\begin{aligned}(I_2 + I_3) \cap C_2 \cap C_3 &= \{(-\eta x, x, y) \mid x, y \in U\} \cap \{(0, -\eta z + z, -z) \mid z \in U\} \\ &= \{(0, x, y) \mid x, y \in U, \eta x = 0, x = \eta y - y\} = 0.\end{aligned}$$

Indeed, $\eta x = 0$ implies that $x = 0$, since η is invertible, and so $\eta y = y$ must hold. By assumption η cannot have eigenvalue 1, so also $y = 0$.

Now we will follow the proof of Proposition 8.2.7 in order to find the hamiltonian vector field associated to (289). We have the symplectic decomposition $V = (I_1 \oplus I_2) \oplus (I_1 \oplus I_2)^{\perp} = (U \times 0 \times U) \oplus (0 \times U \times 0)$. I_3 is the graph of the map $g : (I_1 \oplus I_2)^{\perp} \to I_1 \oplus I_2$ given by

$$0 \times U \times 0 \to U \times 0 \times U, \quad (0, x, 0) \longmapsto (-\eta x, 0, -x),$$

so the image of g is the graph of the map $I_2 \to I_1, (0, 0, x) \mapsto (\eta x, 0, 0)$. With respect to the chosen frame basis, the coordinate matrix of this map is A, and the matrix of the identification of I_1 with I_2^* is T. Thus the image of g corresponds to a map $f : I_2 \to I_2^*$ whose coordinate matrix is TA. Using $TA = (1 - A^t)T$ and that T is skew-symmetric, we find that the antisymmetric part f_a is given by

$$\tfrac{1}{2}(TA - A^t T^t) = \tfrac{1}{2}(TA - T^t(1 - A)) = \tfrac{1}{2}T(A + (1 - A)) = \tfrac{1}{2}T,$$

and for the symmetric part f_s

$$\tfrac{1}{2}(TA + A^t T^t) = \tfrac{1}{2}(TA + T^t(1 - A)) = \tfrac{1}{2}T(A - (1 - A)) = \tfrac{1}{2}T(2A - 1).$$

Thus we obtain the hamiltonian vector field $X = f_a^{-1} f_s$ given in coordinates by the matrix $2A - 1$. We have proved the following:

PROPOSITION 8.5.6. *Non-split isotropic triples satisfy the hypotheses of Proposition 8.2.7: we can construct an associated hamiltonian vector field.*

If φ is such a triple, we may assume that its underlying sextuple is a framed sextuple S_η, built from an endomorphism (U, η). In this case, the symplectic form ω of φ induces an isomorphism $U \to U^*$ which defines a symplectic form ω_U on U. The hamiltonian vector field associated to φ is $(U, \omega_U, 2\eta - id)$.

REMARK 8.5.7. Normal forms for indecomposable linear hamiltonian vector fields over $\mathbb{R}$ are given, for example, in [**LM74**]. These come in both "split" and "non-split" types, and are labeled by their (complex) eigenvalues. For a non-split isotropic triple as above with underlying indecomposable endomorphism η, the associated hamiltonian vector field

$2\eta - \mathrm{id}$ is also indecomposable and non-split. The possible such η are parametrized by $\frac{1}{2} \pm \sqrt{-1} \cdot r$ with $r \in [0, \infty)$; the corresponding hamiltonian vector fields $2\eta - \mathrm{id}$ correspond in [**LM74**] to the non-split ones labeled by complex eigenvalues $\pm\sqrt{-1} \cdot \nu$ (setting $\nu = 2r$ in order to use their notation), where $\nu \in [0, \infty)$.

EXAMPLE 8.5.8. Let $k = 4$, and let η be an indecomposable endomorphism over $\mathbb{R}$ with complex eigenvalue $\frac{1}{2}$, i.e. with respect to a Jordan basis η has the coordinate matrix

$$A = \begin{pmatrix} 1/2 & 1 & 0 & 0 \\ 0 & 1/2 & 1 & 0 \\ 0 & 0 & 1/2 & 1 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}.$$

We can make the corresponding self-dual sextuple $\mathcal{S}_\eta$ into an isotropic triple by choosing the compatible symplectic form given, with respect to a Jordan frame basis, by the matrix H_T, with

$$T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

The associated hamiltonian vector field on the symplectic space $(\mathbb{R}^2, T)$ is given, with respect to a Jordan basis for η, by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

EXAMPLE 8.5.9. Let $k = 4$, and let η be an indecomposable endomorphism over $\mathbb{R}$ with complex eigenvalue $\frac{1}{2} \pm \sqrt{-1}\frac{1}{2}\nu$, where $\nu > 0$, i.e. with respect to a (real) Jordan basis η has the coordinate matrix

$$A = \frac{1}{2}\begin{pmatrix} 1 & -\nu & 1 & 0 \\ \nu & 1 & 0 & 1 \\ 0 & 0 & 1 & -\nu \\ 0 & 0 & \nu & 1 \end{pmatrix}.$$

Again choosing the compatible symplectic form for $\mathcal{S}_\eta$ given by H_T with

$$T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

we obtain the associated hamiltonian vector field on $(\mathbb{R}^2, T)$ given by

$$\begin{pmatrix} 0 & -\nu & 1 & 0 \\ \nu & 0 & 0 & 1 \\ 0 & 0 & 0 & -\nu \\ 0 & 0 & \nu & 0 \end{pmatrix}.$$

8.6. Continuous non-split isotropic triples over perfect fields

We continue working with a ground field $\mathbf{k}$ with $\mathrm{char}(\mathbf{k}) \neq 2$, and additionally we assume that $\mathbf{k}$ is perfect[14], in order that we may use the normal forms discussed in Proposition 8.6.7 below. No other assumptions are made on $\mathbf{k}$.

According to Proposition 8.4.17, isomorphisms of a sextuple $\mathcal{S}_\eta$ onto its dual arise from matrices M such that $M^{-1}AM = (I - A)^t$ where A is the matrix of η with respect to some basis; and the induced compatible form is (skew-)symmetric if and only if M is. As indicated by the results over the complex or real number field, the crucial case is when η has irreducible minimal polynomial $q(x)$. We deal with this case first.

8.6.1. Irreducible characteristic polynomial. Let $q(x) = \sum_{i=0}^{l} a_i x^i \in \mathbf{k}[x]$ be an irreducible polynomial, with $a_l = 1$. Adjunction to $\mathbf{k}$ of a zero λ of q yields an extension field $\mathbf{k}(\lambda) = \mathbf{k}[\lambda] \cong \mathbf{k}[x]/q(x)$ which as a $\mathbf{k}$-vector space has basis $1, \lambda, \lambda^2, \ldots, \lambda^{l-1}$. With respect to this basis, the $\mathbf{k}$-linear map m_λ defined by $m_\lambda(r) := \lambda r$ has as its coordinate matrix the **Frobenius matrix** $N^t + C$

$$\begin{pmatrix} 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & \ddots & 0 & -a_1 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & 0 & -a_{l-2} \\ 0 & \ldots & 0 & 1 & -a_{l-1} \end{pmatrix},$$

where $C = C_q$ denotes the matrix whose last column is $-a_0, -a_1, -a_2, \ldots, -a_{l-1}$ and has all other entries zero.

In particular, by the Cayley-Hamilton theorem the above applies to any endomorphism η of an vector space U having characteristic polynomial $q(x)$, where η plays the role of λ above[15]. Here we view $\mathbf{k}(\eta)$ as a subring of $\mathrm{End}(U)$ with subfield $\{a\mathsf{id} \mid a \in \mathbf{k}\} \cong \mathbf{k}$. In this situation, any $v_1 \neq 0$ extends (uniquely) to a basis $\{v_1, \eta v_1, ..., \eta^{l-1} v_1\}$ such that the corresponding coordinate matrix of η is the Frobenius matrix of $q(x)$. In other words, if $A \in \mathbf{k}^{l\times l}$ has $q(x) = \det(xI - A)$ irreducible, then $\mathbf{k}(A) = \mathbf{k}[A]$ is a subring of $\mathbf{k}^{l\times l}$ which is an extension field of $\{aI \mid a \in \mathbf{k}\} \cong \mathbf{k}$ with primitive element A, a zero of $q(x)$.

Lemma 8.6.1. *Let $q(x) \in \mathbf{k}[x]$ be an irreducible monic polynomial with $\deg q = l > 1$. Let λ be a zero of q in some extension field of $\mathbf{k}$, and set $\mathbf{E} = \mathbf{k}(\lambda)$.*

Suppose $q(1 - \lambda) = 0$. Then there exists $\mu \in \mathbf{E}$ and an automorphism g of $\mathbf{E}$ over $\mathbf{k}$ such that $\mathbf{k}(\mu) = \mathbf{E}$ and $g(\mu) = -\mu$. This implies that:

1. *Only even powers of x occur in the minimal polynomial $r(x)$ of μ over $\mathbf{k}$; in particular $\deg r = l$ must be even.*
2. *With respect to the basis $\{1, \mu, ..., \mu^{l-1}\}$ of $\mathbf{E}$ over $\mathbf{k}$, g has diagonal coordinate matrix D, with entries $d_{ii} = (-1)^i$, for $i = 0, ..., l - 1$.*
3. *g is a $\mathbf{k}$-isomorphism $(\mathbf{E}, m_{-\mu}) \to (\mathbf{E}, m_\mu)$, i.e. $m_\mu g = g m_{-\mu}$.*

Proof. Since $q(\lambda) = q(1 - \lambda) = 0$, there exists an automorphism g of $\mathbf{E} = \mathbf{k}(\lambda)$ over $\mathbf{k}$ such that $g(\lambda) = 1 - \lambda$. Setting $\mu := \lambda - \frac{1}{2}$, we have $\lambda = \frac{1}{2} + \mu$ and $1 - \lambda = \frac{1}{2} - \mu$; in

[14] A field $\mathbf{k}$ is perfect if every algebraic extension of $\mathbf{k}$ is separable. Examples of perfect fields include all finite fields, and all fields of characteristic zero.

[15] Recall that for indecomposable endomorphisms, the characteristic and minimal polynomials coincide. Also note that, when the characteristic polynomial is irreducible, the corresponding endomorphism is necessarily indecomposable.

particular $\mathbf{k}(\lambda) = \mathbf{k}(\mu)$ and $g(\mu) = -\mu$. Note that $-\mu \neq \mu$, since $-\mu = \mu$ would imply $\lambda = 1/2 \in \mathbf{k}$, and hence that $[\mathbf{k}(\lambda) : \mathbf{k}] = \deg(q) = l = 1$, contrary to the assumption $l > 1$. For the minimal polynomial

$$r(x) = a_l x^l + a_{l-1} x^{l-1} + ... + a_1 x + a_0 \qquad (\text{with } a_l = 1)$$

of μ over $\mathbf{k}$ it follows that $r(-\mu) = r(g(\mu)) = g(r(\mu)) = 0$, using that g fixes $\mathbf{k}$. This implies that only even powers of x occur in $r(x)$, by comparing the coefficients of $r(-\mu)$ and $r(\mu)$. Indeed,

$$\sum_{k \text{ odd}} a_k \mu^k = \frac{1}{2}(r(\mu) - r(-\mu)) = 0$$

since $r(\mu) = r(-\mu) = 0$, so μ is a zero of $\delta(x) := \sum_{k \text{ odd}} a_k x^k$, and hence $r(x) \mid \delta(x)$. If $r(x)$ were to have odd degree, then $\delta(x)$ and $r(x)$ would have the same degree and hence would be equal (since they are both monic). But this would contradict the irreducibility of $r(x)$, because for a polynomial with only odd powers of x, one can always factor out the polynomial $p(x) = x$. Therefore it must be that $\delta(x) \equiv 0$, i.e. $a_k = 0$ for all odd k.

The statement about the coordinate matrix of g is clear. For point 3., we check on the basis elements μ^i of $\mathbf{E}$:

$$m_\mu(g(\mu^i)) = m_\mu((-1)^i \mu^i) = (-1)^i \mu^{i+1} = -(-1)^{i+1} \mu^{i+1} = g(m_{-\mu}(\mu^i)).$$

□

PROPOSITION 8.6.2. *Let $0 \neq A \in \mathbf{k}^{l \times l}$ such that $q(x) = \det(xI - A)$ is irreducible. Then the following are equivalent:*

1. *A and $(I - A)^t$ are similar to each other*
2. *A and $I - A$ are similar to each other*
3. *$q(\lambda) = q(1 - \lambda) = 0$ for some λ in some extension field of $\mathbf{k}$.*
4. *For any λ in any extension field of $\mathbf{k}$: if $q(\lambda) = 0$, then $q(1 - \lambda) = 0$.*

Suppose now that any of these equivalent conditions holds. Then $\deg q(x)$ is even. Moreover, setting $\tilde{A} = A - \frac{1}{2}I$, the minimal polynomial $r(x)$ of $\tilde{A}$ satsifies $r(-\tilde{A}) = 0$ and, for any invertible T,

$$TAT^{-1} = (I - A)^t \;\Leftrightarrow\; T\tilde{A}T^{-1} = -\tilde{A}^t.$$

PROOF. That 1. and 2. are equivalent follows from the fact that a matrix and its transpose are always similar, and that similarity is a transitive relation. 2. implies that $q(I - A) = 0$, and thus that 3. holds in the field $\mathbf{k}(A) = \mathbf{k}[A] \in \mathbf{k}^{l \times l}$. To see that 3. implies 4., it is enough to consider extensions $\mathbf{k}(\lambda)$ of $\mathbf{k}$ such that $q(\lambda) = 0$; these are all isomorphic via isomorphisms fixing $\mathbf{k}$, and hence in each of them also the equation $q(1 - \lambda) = 0$ is satisfied. Finally we show that 4. implies 2.. Since $q(A) = 0$, 4. implies $q(I - A) = 0$ and thus that $q(x)$ is also the unique elementary divisor of $I - A$. The multiset of elementary divisors is a complete invariant for similarity, so 2. follows. That $\deg q$ is even and that $r(x)$ satisfies $r(-\tilde{A})$ follows from Lemma 8.6.1. □

REMARK 8.6.3. In the subsequent, we'll use the following fact. Let $\mathbf{E} \supseteq \mathbf{k}$ be a finite field extension, so $\mathbf{E}$ is a finite-dimensional vector space over $\mathbf{k}$, and let $\tau : \mathbf{E} \to \mathbf{k}$ be a non-zero $\mathbf{k}$-linear map. Then the bilinear form on $\mathbf{E}$ defined by

$$(u, v) \longmapsto \tau(uv)$$

is non-degenerate. Indeed, if for fixed $v \in \mathbf{E}\backslash\{0\}$ we have $\tau(uv) = 0$ for all $u \in \mathbf{E}$, then it would follow that $\tau(u') = 0$ for all $u' \in \mathbf{E}$, since any u' can be written as $u'v^{-1}v$. This would contradict the assumption that $\tau \neq 0$.

PROPOSITION 8.6.4. *Let $l > 0$, and let $0 \neq A \in \mathbf{k}^{l\times l}$ be a Frobenius matrix with irreducible characteristic polynomial $r(x)$ such that $r(-A) = 0$. Then there are skew-symmetric as well as symmetric invertible $T \in \mathbf{k}^{l\times l}$ such that $TAT^{-1} = -A^t$.*

PROOF. We may restate the task as follows. We are given $V = \mathbf{k}(\mu)$, $\dim V = l$ even, with irreducible monic $r(x) = \sum_{i=0}^{l} a_i x^i$ such that $r(\mu) = r(-\mu) = 0$, in particular $a_i = 0$ for odd i. Let $m_\mu(v) = \mu v$ for $v \in V$. The task is to find an isomorphism $\beta : V \to V^*$ such that

$$\beta m_\mu = -m_\mu^* \beta, \tag{290}$$

and such that the matrix T of β with respect to the basis $1, \mu, \dots, \mu^{l-1}$ and its dual basis $1^*, \mu^*, \dots$ is skew-symmetric resp. symmetric. Because of these bases, within the scope of this proof we use indices which always live between 0 and $l-1$.

We consider the skew-symmetric task first. From [**QSS79**], page 276, we know there exists symmetric S such that $SAS^{-1} = A^t$. Namely, one has the linear form $\tau \in V^*$ defined on the basis $1, \mu, \dots, \mu^{l-1}$ by

$$\tau(\mu^i) = 1 \;\text{ if } i = l-1, \quad \tau(\mu^i) = 0 \text{ else}$$

and we take S to be the matrix of the non-degenerate symmetric bilinear form $(u, v) \mapsto \tau(uv)$, i.e. the entries of S are

$$s_{ij} = \tau(\mu^{i+j}) \qquad 0 \le i, j \le l-1.$$

Let V_0 be the $\mathbf{k}$-subspace of V spanned by $\{\mu^i \mid 0 \le i < l,\ i \text{ even}\}$; let V_1 be the $\mathbf{k}$-subspace spanned by $\{\mu^i \mid 0 \le i < l,\ i \text{ odd}\}$.

CLAIM 8.6.5. For any $i \in \mathbb{N}$, $\mu^i \in V_0$ if i is even, $\mu^i \in V_1$ if i is odd. In particular $\tau(\mu^i) = 0$ for all even $i \in \mathbb{N}$.

Proof of the claim. We use induction. Clearly, the claim holds for $i < l$. Also, note that $\mu^l = -\sum_{j<l} a_j \mu^j$ where $a_j = 0$ if j is odd, and so $\mu^l \in V_0$. In particular, we see that $\mu V_0 \subseteq V_1$ and $\mu V_1 \subseteq V_0$. Now let $i > l$. If i is odd, then by the induction hypothesis $\mu^{i-1} \in V_0$, and so $\mu^i = \mu\mu^{i-1} \in \mu V_0 \subseteq V_1$. The analogous argument applies for i even. □

It follows that S has an following properties: using indices $i, j \in \{0, \dots, l-1\}$, the entries s_{ij} of S only depend on $i + j$, with $s_{ij} = 0$ for $i + j \le l - 2$ or $i + j$ even, and $s_{ij} = 1$ for $i + j = l - 1$.

Now consider $T := SD$, where D is the diagonal matrix defined in Lemma 8.6.1, i.e. the diagonal entries are $d_{ii} = (-1)^i$ for $i = 0, ..., l-1$. Thus T has entries $t_{ij} = s_{ij}$ if j is even, $t_{ij} = -s_{ij}$ if j is odd. In particular then, $t_{ij} = 0$ if $i+j$ is even (since $s_{ij} = 0$ if $i+j$ is even) and $t_{ij} = -t_{ji}$ if $i + j$ is odd (since i and j must have different parity when $i + j$ is odd). So T is skew symmetric.

With respect to the basis $1, \mu, \mu^2, \dots$ and its dual basis, S provides (according to [**QSS79**] Prop. 4.4) an isomorphism $(V, m_\mu) \to (V^*, m_\mu^*)$, while by Lemma 8.6.1 D provides an isomorphism $(V, -m_\mu) \to (V, m_\mu)$. Thus $T = SD$ gives an isomorphism $(V, -m_\mu) \to (V^*, m_\mu^*)$, as desired.

With the skew-symmetric case done, we now turn to the symmetric case. Recall that we want to find β such that (290) holds; for this we work with the coordinate matrix T of β with respect to the basis $1, \mu, \dots, \mu^{l-1}$ and its dual basis.

For convenience, we write $T = (t_{ij})_{i,j=0,\dots,l-1}$. Summations will be for $0, \dots, l-1$ unless stated otherwise.

CLAIM 8.6.6. Equation (290) holds for T if and only if all of the following are satisfied:

(1) $t_{k,j+1} = -t_{k+1,j}$ for $j, k < l-1$
(2) $\sum_h -a_h t_{kh} = -t_{k+1,l-1}$ for $j = l-1,\ k < l-1$
(3) $t_{l-1,j+1} = \sum_h a_h t_{hj}$ for $j < l-1,\ k = l-1$
(4) $-\sum_h a_h t_{l-1,h} = \sum_h a_h t_{h,l-1}$ for $j = k = l-1$

Proof of Claim. Let $j, k \le l-1$. The matrix entries corrsponding to $\beta\hat{\mu}$ and $-\hat{\mu}^*\beta$, respectively, are

$$x_{jk} :=(\beta\hat{\mu}(\mu^j))(\mu^k) = (\beta(\mu^{j+1}))(\mu^k)$$
$$y_{jk} :=(-\hat{\mu}\beta(\mu^j))(\mu^k) = -\sum_i t_{ij}\hat{\mu}^*(\mu^{i*})(\mu^k) = -\sum_i t_{ij}\mu^{i*}(\hat{\mu}(\mu^k)) = -\sum_i t_{ij}\mu^{i*}(\mu^{k+1}),$$

and (290) holds if and only if $x_{jk} = y_{jk}$ for all $j, k \le l-1$. Now,

$$x_{jk} = \sum_i t_{i,j+1}\mu^{i*}(\mu^k) = t_{k,j+1} \quad \text{if } j < l-1$$
$$y_{jk} = -\sum_i t_{ij}\mu^{i*}(\mu^{k+1}) = -t_{k+1,j} \quad \text{if } k < l-1.$$
$$x_{l-1,k} = \beta(\mu^l)(\mu^k) = \beta(\sum_h -a_h\mu^h)(\mu^k) = \sum_{i,h} -a_h t_{ih}\mu^{i*}(\mu^k) = \sum_h -a_h t_{kh}$$
$$y_{j,l-1} = -\sum_i t_{ij}\mu^{i*}(\mu^l) = -\sum_i t_{ij}\mu^{i*}(\sum_h -a_h\mu^h) = \sum_{i,h} t_{ij}a_h\mu^{i*}(\mu^h) = \sum_h t_{hj}a_h$$

Thus, that $x_{jk} = y_{jk}$ for all j, k is equivalent to the equations stated in the claim. □

Define $\tilde{\tau} \in V^*$ by

$$\tilde{\tau}(\mu^i) = 1 \text{ if } i = l-2, \quad \tilde{\tau}(\mu^i) = 0 \text{ else}$$

(note that we use that $l > 0$). By Remark 8.6.3, the matrix $\tilde{S}$ whose entries are $\tilde{s}_{ij} = \tilde{\tau}(\mu^{i+j})$ is invertible. Note that $\tilde{\tau}(\mu^k) = 0$ for odd $k \in \mathbb{N}$, since $\tilde{\tau}|_{V_1} = 0$. In particular, $(-1)^{i+1}\tilde{\tau}(\mu^{i+j}) = (-1)^{j+1}\tilde{\tau}(\mu^{i+j})$ for all $0 \le i, j \le l-1$, since both sides are zero when i and j have different parity. Thus, if we define

$$t_{ij} := (-1)^{i+1}\tilde{\tau}(\mu^{i+j}) = (-1)^{j+1}\tilde{\tau}(\mu^{i+j})$$

we obtain a symmetric matrix T. Note that $T = -\tilde{S}D$, so T is invertible. The entries t_{ij} only depend on $i+j$, up to sign. For for $i+j < l-2$ they are zero; for $i+j = l-2$, $t_{ij} = (-1)^i = (-1)^j$.

It remains now only to check that this T fulfills the conditions of Claim 8.6.6:

(1) $t_{k,j+1} = (-1)^{k+1}\tilde{\tau}(\mu^{k+j+1}) = -t_{k+1,j}$ for $j, k < l-1$
(2) $t_{k+1,l-1} = (-1)^k\tilde{\tau}(\mu^{k+l}) = (-1)^k\sum_h -a_h\tilde{\tau}(\mu^{k+h}) = \sum_h a_h(-1)^{k+1}\tilde{\tau}(\mu^{k+h}) = \sum_h a_h t_{kh}$
for $j = l-1,\ k < l-1$
(3) $t_{l-1,j+1} = (-1)^j\tilde{\tau}(\mu^{j+l}) = (-1)^j\sum_h -a_h\tilde{\tau}(\mu^{j+h}) = \sum_h a_h(-1)^{j+1}\tilde{\tau}(\mu^{j+h}) = \sum_h a_h t_{hj}$
for $j < l-1,\ k = l-1$

(4) $-\sum_h a_h t_{l-1,h} = -\sum_{h \text{ even}} a_h t_{l-1,h} = 0$ since $l-1+h$ is odd for even h.
Similarly, $\sum_h a_h t_{h,l-1} = \sum_{h \text{ even}} a_h t_{h,l-1} = 0$ since $l-1+h$ is odd for even h.

□

8.6.2. Generalized Jordan blocks. In this section we recall some well-known facts about normal forms for endomorphisms. Let $m, l \in \mathbb{N}$ be natural numbers, and consider $ml \times ml$-matrices A with $l \times l$-blocks A_{ij}. Let N be the standard nilpotent matrix of size ml. Then N^l has blocks I_l as first upper off-diagonal of blocks, and all other entries zero, i.e.

$$N^l = \begin{pmatrix} O & I_l & O & \\ O & O & I_l & \\ & \ddots & \ddots & \ddots \end{pmatrix}.$$

Note that for any invertible matrix $K \in \mathbf{k}^{l \times l}$, one has $(KI)N^l(K^{-1}I) = N^l$, where

$$KI = \begin{pmatrix} K & O & O & \\ O & K & O & \\ & \ddots & \ddots & \ddots \end{pmatrix}.$$

Proposition 8.6.7. *Let η be an indecomposable endomorphism of the n-dimensional vector space V over a perfect field $\mathbf{k}$. Then*

(1) *There is unique $m \in \mathbb{N}$ and irreducible monic $q(x) \in \mathbf{k}[x]$ such that $q(x)^m$ is the minimal (= characteristic) polynomial of η; that is, $q(x)^m$ is the unique elementary divisor of η. In particular, $ml = n$ where $l = \deg q(x)$. Moreover, as an $\mathbf{k}[x]$-module, V is isomorphic to $\mathbf{k}[x]/q(x)^m$; in particular it is cyclic.*
(2) *Let $q(x)^m$ be the unique monic elementary divisor of η. There is a basis $\bar{v}$ of V with respect to which η has matrix*

$$A = ZI + N^l = \begin{pmatrix} Z & I_l & O & \cdots \\ O & Z & I_l & \ddots \\ O & O & Z & \ddots \\ \vdots & & \ddots & \ddots \end{pmatrix}.$$

where $q(x) = \det(xI_i - Z)$. Conversely, for any basis with respect to which the coordinate matrix of η is of the form $A = ZI + N^l$, one has $\det(xI_l - Z) = q(x)$ and $\det(xI - A) = q(x)^m$.

Proof. 1. This follows from the theory of a linear operator on a finite-dimensional vector space. In general, the $k[x]$-module V is isomorphic to the direct sum of the $\mathbf{k}[x]/d_i(x)$ where the $d_i(x) = q_i(x)^{m_i}$ are the elementary divisors of η, the $q_i(x)$ being coprime irreducibles. The characteristic polynomial is $\prod_i d_i(x) = \det(xI - A)$. That η is indecomposable means that there is only a single elementary divisor $q(x)^m$, and so the characteristic polynomial and minimal polynomial of A coincide with $q(x)^m$.

2. For the existence of such a basis and corresponding normal form we use our assumption that $\mathbf{k}$ is perfect; see [**Mal63**] for a discussion of this normal form, and [**Rob70**] and [**Dal14**] regarding necessary and sufficient conditions for its existence. Conversely, given such a normal form for η, it follows from the behaviour of block matrices and determinants that $q(x)^m = \det(xI - A) = (\det(xI_l - Z))^m$. By the uniqueness of factorization of monic polynomials into irreducible polynomials, and by comparing degrees, it follows that $\det(xI_l - Z) = q(x)$. □

REMARK 8.6.8. We refer to the normal form given in part 2 above as "generalized Jordan normal form". The matrix Z is, in coordinate form, a "generalized eigenvalue" of η.

8.6.3. Admissible forms for self-duals.

THEOREM 8.6.9. *Given an indecomposable sextuple $\mathcal{S}_\eta$ with no eigenvalue in* $\mathbf{k}$*, the following statements are equivalent:*

(1) *it is self-dual*
(2) *it admits compatible symplectic forms*
(3) *it admits compatible symmetric forms.*

In the case of self-duality, compatible ε-symmetric forms can be given, with respect to suitable bases, by matrices of the form

$$H_M = \begin{pmatrix} O & O & -M \\ O & M & O \\ -M & O & O \end{pmatrix}, \qquad \textit{with } M = \begin{pmatrix} \vdots & \vdots & & & \\ O & O & T & & \\ O & -T & O & \cdots & \\ T & O & O & \cdots & \end{pmatrix} \in \mathbf{k}^{ml\times ml},$$

where $T \in \mathbf{k}^{l\times l}$ and $q(x)^m$ is the characteristic polynomial of η, with $q(x)$ irreducible and $\deg q(x) = l$*. Clearly, $\varepsilon(H_M) = \varepsilon(M) = (-1)^{m+1}\varepsilon(T)$. Suitable bases are frame bases $\{\bar{u}, \bar{v}, \bar{w}\}$ where the basis $\bar{v} = v_1, \ldots, v_{ml}$ renders η in normal form $(\frac{1}{2}I_l + \tilde{Z})I + N^l$ with Frobenius matrix $\tilde{Z} \in \mathbf{k}^{l\times l}$. Here, T can be chosen according to Proposition 8.6.4.*

PROOF. Given compatible forms, self-duality is obvious. Conversely, consider self-dual indecomposable $\mathcal{S}_\eta$ and consider a basis $\bar{v}$ such that the corresponding matrix of η is $A = ZI + N^l$ as in 2. of Proposition 8.6.7. Let $q(x)^m$ be the unique elementary divisor of η, where q is irreducible over $\mathbf{k}$ with $\deg q = l$.

By Proposition 8.4.17, A is similar to $(I - A)^t$. Thus, $q(x)^m$ is also the unique elementary divisor of $(I - A)^t$, implying in particular that $\det(xI_l - (I_l - Z)^t) = q(x)$ (c.f. the proof of part 2. in Proposition 8.6.7). So Z and $(I_l - Z)^t$ share $q(x)$ as their unique elementary divisor, and are hence similar. Now by Proposition 8.6.2, $\tilde{Z} = Z - \frac{1}{2}I_l$ has irreducible characteristic polynomial $r(x)$ with $r(-\tilde{Z}) = 0$.

Since r is irreducible, there exists invertible K such that $K\tilde{Z}K^{-1}$ is Frobenius. Note that $KZK^{-1} = \frac{1}{2}I_l + K\tilde{Z}K^{-1}$ and $KAK^{-1} = \frac{1}{2}I + K\tilde{Z}K^{-1}I + N^l$. Thus we may assume directly that η has matrix $A = ZI + N^l$ with $Z = \frac{1}{2}I_l + \tilde{Z}$ where $\tilde{Z}$ is a Frobenius matrix with irreducible characteristic polynomial $r(x)$ satisfying $r(-\tilde{Z}) = 0$. Moreover, $TZT^{-1} = (I_l - Z)^t$ if and only if $T\tilde{Z}T^{-1} = -\tilde{Z}^t$ for any invertible T (c.f. Proposition 8.6.2). By Proposition 8.6.4, there exist T, skew-symmetric as well as symmetric, which satisfy $T\tilde{Z}T^{-1} = -\tilde{Z}^t$. Choose such a T and let M be defined as above in the statement of this proposition. By Proposition 8.4.17, it suffices to show that $MAM^{-1} = (I - A)^t$.

For this, observe first that $M = TQJ = QTJ$ where J is block-diagonal with diagonal blocks $J_{jj} = (-1)^{j+1}I_l$ and Q is block-anti-diagonal with blocks $Q_{ij} = I_l$ for $i+j = m+1$, and all other entries being 0. Also, observe that $Q^2 = I$ and that Q acts as permutation of blocks: acting on the right it exchanges block-columns j and $m+1-j$, acting on the left it exchanges block-rows i and $m+1-i$. Furthermore, the following properties are easily seen:

- $QDQ = D$ if D is block-diagonal of the form XI for some $X \in \mathbf{k}^{l\times l}$
- $QN^lQ = (N^l)^t$

- $J^2 = I$, and $JDJ = D$ if D is block-diagonal
- $JN^lJ = -N^l$
- $T(N^l)^tT^{-1} = (N^l)^t$ since $(N^l)^t$ has only unit and zero blocks.

It follows that

$$MZIM^{-1} = QTJZIJT^{-1}Q = QTZIT^{-1}Q = Q(I_l - Z)^tIQ = (I_l - Z)^tI$$
$$MN^lM^{-1} = TQJN^lJQT^{-1} = TQ(-N^l)QT^{-1} = -T(N^l)^tT^{-1} = -(N^l)^t$$

and hence

$$MAM^{-1} = MZIM^{-1} + MN^lM^{-1} = (I_l - Z^t)I - (N^l)^t = I - A^t,$$

as desired. □

COROLLARY 8.6.10. *Up to isomorphism, the indecomposable sextuples underlying non-split continuous-type isotropic triples are paremetrized by indecomposable linear endomorphisms η of the form*

$$\eta = \tfrac{1}{2} + \zeta$$

where ζ is an indecomposable endomorphism which lives in an even-dimensional space and is such that ζ is similar to $-\zeta^*$. Such endomorphisms ζ are themselves parametrized by the set

$$\{r^m \mid m \in \mathbb{Z}_{>0},\ r \text{ monic irreducible and } r = p(x^2) \text{ with } p \in \mathbf{k}[x]\} \cup \{(x^2)^m \mid m \in \mathbb{Z}_{>0}\}.$$

PROOF. From Proposition 8.4.17 we know that (isomorphism classes of) indecomposable self-dual continuous-type sextuples are in one-to-one correspondence with (isomorphism classes of) indecomposable endomorphisms η such that η is similar to $1-\eta^*$. Setting $\zeta := \frac{1}{2} - \eta$ it easily seen that ζ is indecomposable if and only if η is, and that

$$\eta \text{ similar to } 1 - \eta^* \quad \Leftrightarrow \quad \zeta \text{ similar to } -\zeta^*.$$

Moreover, Theorem 8.6.9 and Theorem 8.5.1 imply that the underlying sextuples of non-split isotropic triples, up to isomorophism, are parametrized by all endomorphisms η as above, with the exception of those which have an eigenvalue in the ground field and live in an odd-dimensional space.

In the case when η has no eigenvalue in the ground field, it follows from the above proof of Theorem 8.6.9 and from Lemma 8.6.1 that the corresponding endomorphism ζ has a minimal polynomial of the form $r(x)^m$, with r irreducible and with only even powers of the variable appearing, i.e. $r(x) = p(x^2)$ for some $p \in \mathbf{k}[x]$. (The irreducible polynomial r here is the minimal polynomial of the "eigenvalue" $\tilde{Z}$ of ζ which appears in the proof of Theorem 8.6.9; for Lemma 8.6.1 we use $\tilde{Z}$ in the role of μ).

In the case when η does have an eigenvalue in the ground field, the corresponding endomorphism ζ is nilpotent, and hence, if it lives in an even-dimensional space, it has a minimal polynomial of the form $r(x) = (x^2)^m$ for some $m \in \mathbb{Z}_{>0}$ □

8.6.4. Endomorphism algebras. In order to address the question of uniqueness of compatible forms, we first recall the structure of endomorphism algebras of indecomposable sextuples $\mathcal{S}_\eta$. We've seen that these are isomorphic to endomorphism algebras of vector spaces with indecomposable endomorphism, i.e. endomorphism algebras of $\mathbf{k}[x]$-modules $V_{\mathbf{k}[\eta]}$, where η is an indecomposable endomorphism of a vector space V over $\mathbf{k}$. These are well-known; we recall some basic facts.

PROPOSITION 8.6.11. *Given $\eta, Z, A, \bar{v}$ as in Proposition 8.6.7, the following hold:*

(1) *The endomorphism algebra E of $V_{\mathbf{k}[\eta]}$ is isomorphic to the $\mathbf{k}$-algebra $\mathbf{k}[x]/q(x)^m$, which is of dimension $n = ml = \dim_{\mathbf{k}} V$, where l is the degree of the irreducible minimal polynomial $q(x)$ of η. In particular, E is local with radical $E \cdot q(x)$.*
(2) *With respect to $\bar{v}$, E is given by the matrices C such that $CA = AC$. We have $C \in E$ if and only if $C = \sum_{i=0}^{k-1} Z_i N^{li}$ with $Z_i \in \mathbf{k}(Z)$, and this representation is unique. In particular, C is upper block triangular with diagonal blocks $C_{11} = \ldots = C_{kk} = Z_0$ and C is invertible in E if and only if $Z_0 \neq 0$.*

PROOF. 1. Rephrasing the definitions, E is the collection of $\mathbf{k}$-linear endomorphisms of the vector space V which commute with η. For η indecomposable, E is simply $\mathbf{k}[\eta] \subseteq \mathrm{End}(V)$, and hence is isomorphic to $\mathbf{k}[x]/q(x)^m$. One way to see this is to note that since η is indecomposable, there exists a basis of V of the form $\{v, \eta v, ..., \eta^{n-1}v\}$ for some $v \in V$. For any $f \in E$, observe that $fv = \sum_{i=0}^{n-1} c_i\eta^i v$ for some $c_i \in \mathbf{k}$; it follows then that in fact $f = \sum_{i=0}^{n-1} c_i\eta^i$ since the latter expression coincides with f on the above given basis of V.

2. The first statement is clear. To prove the first "if" it suffices to observe that, for $W \in \mathbf{k}(Z)$, one has $WZ = ZW$ since $\mathbf{k}(Z)$ is commutative and $WIN^l = N^l WI$ since N^l has zero and unit blocks only. Thus, matrices $C = \sum_{i=0}^{k-1} Z_i N^{li}$ form a subalgebra E' of E.

To prove the "only if", we show that E' and E have the same dimension over $\mathbf{k}$. For this, we compute the dimension of E' as an $\mathbf{k}(Z)$-algebra. Since $N^{lm} = 0$, induction on $m - h > 0$ shows that $I, N^l, N^{l2}, \ldots, N^{l(m-1)}$ are independent over $\mathbf{k}(Z)$: if $\sum_{j=h}^{m-1} Z_j N^{lj} = 0$ then, applying N^l, we get $\sum_{j=h}^{m-2} Z_j N^{l(j+1)} = 0$ and hence, using the induction hypothesis, that $Z_h = ... = Z_{m-1} = 0$. Thus, E' has dimension m over $\mathbf{k}(Z)$ and so dimension $ml = n = \dim E$ over $\mathbf{k}$, since $\mathbf{k}(Z)$ has dimension l over $\mathbf{k}$. Thus, $E' = E$. □

8.6.5. Uniqueness: General result. In this section we generalize Lemma 7.8.1 in a way which applies to all poset representations and also to endomorphisms (V, η). Recall (see Section 7.2) that for the latter there is a natural notion of direct sum, morphism, etc., and that for an indecomposable endomorphism (V, η), its endomorphism algebra

$$\mathrm{End}(V, \eta) = \{f : V \to V \mid f\eta = \eta f\}$$

is always local. For our application, we will use the following notion of the **dual** of an endomorphism: we define

$$(V, \eta)^* := (V^*, \mathrm{Id} - \eta^*).$$

Analogous to the definition for poset representation, a compatible form for (U, η) is an isomorphism $B : (V, \eta) \to (V^*, \mathrm{Id} - \eta^*)$ which defines either to a symmetric or skew-symmetric bilinear form on V.

Within the current subsection, the word "representation" and the notation "ψ" will be used to denote a poset representation or an endomorphism (unless further specification is given) and similarly for compatible forms, etc.. Given a bilinear form B we write $\varepsilon(B) = 1$ if B is symmetric, and $\varepsilon(B) = -1$ if B is skew-symmetric. Similar notation also applies for matrices, we set $A^{-t} = (A^{-1})^t$, and we use $A^* = A^t$ interchangeably.

Consider now the following situation. Assume that we are given an indecomposable representation ψ in V of dimension ml, a basis of V, and a subfield F of $\mathbf{k}^{l \times l}$ such that the endomorphism algebra E of ψ is given by block matrices (with $l \times l$-blocks) of the form

$$\sum_{i=0}^{m-1} Z_i N^{li}$$

with unique $Z_i \in F$ and such that $ZN^l = N^l Z$ for all $Z \in F$. In particular, E is commutative and has radical $\mathsf{rad}E = EN^l = \sum_{i=1}^{m-1} X_i N^{li}$.

Further, assume that we are given a compatible form for ψ, which we call B_1. Denote the corresponding coordinate matrix by H_1, which is itself necessarily also a block matrix as above.

DEFINITION 8.6.12. *Let ψ, F, H_1 be as above. For $A \in \mathbf{k}^{l\times l}$, set $A^\dagger := H_1^{-1} A H_1$. The anti-involution "$(-)^\dagger$" restricts to one on F via $Z^\dagger := (ZI)^\dagger$. We define*

$$F_{H_1}^+ := \{Z \in F \mid Z^\dagger = Z\} \qquad\qquad F_{H_1}^- := \{Z \in F \mid Z^\dagger = -Z\}.$$

Note that

$$F_{H_1}^+ = \{Z \in F \mid Z^t H_1 = H_1 Z\} \qquad\qquad F_{H_1}^- = \{Z \in F \mid Z^t H_1 = -H_1 Z\}.$$

REMARK 8.6.13. $F = F_{H_1}^+ \oplus F_{H_1}^-$ by the usual argument: any $Z \in F$ has a unique decomposition

$$Z = \tfrac{1}{2}(Z + Z^\dagger) + \tfrac{1}{2}(Z - Z^\dagger)$$

into selfadjoint and anti-selfadjoint parts.

REMARK 8.6.14. If $0 \neq Z \in F_{H_1}^+$, then $H := H_1 Z$ determines a compatible form B such that $\varepsilon(B) = \varepsilon(B_1)$, since in this case

$$(H_1 Z)^t = Z^t H_1^t = \varepsilon_1 Z^t H_1 = \varepsilon_1 H_1 Z.$$

Similarly, given $0 \neq Z' \in F_{H_1}^-$, then $H' := H_1 Z$ defines an compatible form B' such that $\varepsilon(B') = -\varepsilon(B_1)$.

LEMMA 8.6.15. *Consider, in addition to B_1, another compatible form B_2 for ψ, with associated coordinate matrix H_2. Set $\varepsilon_1 := \varepsilon(B_1)$, $\varepsilon_2 := \varepsilon(B_2)$ and $\epsilon = \varepsilon_1\varepsilon_2$.*

If $\epsilon = 1$, there exists $0 \neq Z \in F_{H_1}^+$ and an automorphism f of ψ which is an isometry from B_2 to the compatible form given by $H_1 Z$.

Notation: In this case we say that B_2 and B_1 are *equivalent up to automorphisms of ψ and multiplication with "scalars" in F.*

PROOF. Let † denote the antiautomorphism given by the operation of adjoint with respect to H_1, i.e. $A^\dagger = H_1^{-1} A^t H_1$. Note that when A is in the endomorphism algebra E of ψ, then so is $A^\dagger$. Note also that $(H_1^{-1})^\dagger = H_1^{-t}$.

Observe that $H_1^{-1} H_2$ determines an automorphism of ψ, so

$$H_1^{-1} H_2 = C_0 I - R_0 \text{ for some invertible } C_0 \in F \text{ and } R_0 \in \mathsf{rad}E.$$

It follows that

$$\begin{aligned}(C_0 I)^\dagger - R_0^\dagger = (H_1^{-1} H_2)^\dagger = H_2^\dagger (H_1^{-1})^\dagger &= H_1^{-1} H_2^t H_1 H_1^{-t} \\ &= H_1^{-1} \varepsilon_2 H_2 H_1 \varepsilon_1 H_1^{-1} = \epsilon H_1^{-1} H_2 = \epsilon C_0 I - \epsilon R_0.\end{aligned}$$

Since $E = FI \oplus \operatorname{rad} E$ and this decomposition is preserved under taking adjoints, $(C_0 I)^\dagger = \epsilon C_0 I$ and $R_0^\dagger = \epsilon R_0$; in particular $C_0 \in F_{H_1}^\epsilon$.

Let $\epsilon = 1$. Since $A \mapsto A^\dagger$ is an anti-automorphism of E, $(C_0^{-1} I)^\dagger = \epsilon C_0^{-1} I = C_0^{-1} I$. Set

$$R := C_0^{-1} R_0 = R_0 C_0^{-1}, \quad H_3 := H_2 C_0^{-1}, \quad C := H_1^{-1} H_3 = I - R$$

Then $R^\dagger = (C_0^{-1})^\dagger R_0^\dagger = C_0^{-1} R_0 = R$ and since R is nilpotent we can proceed as in Lemma 7.8.1 and construct a unit $h \in E$ such that $h^* H_1 h = H_3$ (where H_3 here plays the role of H_2 in that Lemma). Setting $f := h^{-1}$ and using that E is commutative, we obtain

$$f^* H_2 f = f^* H_3 C_0 f = f^* H_3 f C_0 = H_1 C_0.$$

□

LEMMA 8.6.16. *Let B_1 be as above, and assume that its matrix H_1 is zero above the $l \times l$-block anti-diagonal. Let B_0 be another compatible form, with matrix $H_1 Z$ for some non-zero $Z \in F$.*

There is an automorphism f of ψ which is an isometry from B_0 to B_1 if and only if there exist $X \in F^+_{H_1}$ and $Y \in F^-_{H_1}$ such that $Z = X^2 - Y^2$.

PROOF. If $Z = X^2 - Y^2$ with $X \in F^+_{H_1}$ and $Y \in F^-_{H_1}$ then define f by $(X+Y)I$. In this case,

$$\begin{aligned} f^* H_1 Z f = (X+Y)^t H_1 Z (X+Y) &= (X^t + Y^t) H_1 (X+Y) Z \\ &= (X^t + Y^t)(X^t - Y^t) H_1 Z = Z^t H_1 Z = H_1, \end{aligned}$$

using in the last step that $Z = X^2 - Y^2$ implies that $Z \in F^+_{H_1}$.

Conversely, let $f \in \mathrm{End}(\psi)$ be an isometry from B_0 to B_1. The matrix A of f is of the form $A = \sum_i Z_i N^i$ with $Z_i \in F$ and $A^t H_1 A = H_1 Z$. Observe that N commutes with the Z_i and that applying N on the right of a matrix moves all columns of blocks one step to the right (with overspill) and sets the first column to zero. Similarly, application of N^t on the left shifts block rows downward. Thus, since H_1 is zero above the block anti-diagonal, $(N^t)^j H_1 N^i$ has zero block anti-diagonal if $i + j > 0$. Now

$$H_1 Z = A^t H_1 A = \sum_{i,j} Z_j^t (N^t)^j H_1 N^i Z_i$$

and $H_1 Z$ is zero above the block anti-diagonal, and on the block anti-diagonal the blocks are

$$T_{ij} Z = Z_0^t T_{ij} Z_0, \quad \text{with } i + j = m + 1$$

where the T_{ij} denote the blocks of H_1. By hypothesis, $Z_0 = X + Y$ for some $X \in F^+_{H_1}$ and $Y \in F^-_{H_1}$. It follows that

$$Z = T_{ij}^{-1} (X^t + Y^t) T_{ij} (X+Y) = (X-Y)(X+Y) = X^2 - Y^2.$$

□

8.6.6. Uniqueness for sextuples $\mathcal{S}_\eta$. For indecomposable self-dual framed sextuples $\mathcal{S}_\eta$ we obtain the following uniqueness result for compatible forms by applying Lemma 8.6.15 and Lemma 8.6.16 to the underlying endomorphism (U, η). minimal polynomial

THEOREM 8.6.17. *Let $\mathcal{S}_\eta$ be an indecomposable self-dual framed sextuple such that η has no eigenvalue in $\mathbf{k}$, and let $q(x)^m$ be the minimal polynomial of η, with $q(x)$ irreducible, $\deg q(x) = l$.*

Let B_1 and B_2 be compatible (skew)symmetric forms for $\mathcal{S}_\eta$. Set $\varepsilon_1 := \varepsilon(B_1)$, $\varepsilon_2 := \varepsilon(B_2)$, and $\epsilon := \varepsilon_1 \varepsilon_2$. Furthermore:

- *choose a basis of U which is as in Proposition 8.6.7 and extend this to a frame basis for $\mathcal{S}_\eta$;*
- *define a subfield $F \subseteq \mathbf{k}^{l \times l}$ by $F = \mathbf{k}(Z)$, with Z from Proposition 8.6.7.*

In terms of the respective coordinate matrices H_1 and H_2 of the compatible forms B_1 and B_2, we have:

(1) *If $\epsilon = 1$, then H_1 and H_2 are equivalent up to automorphisms of $\mathcal{S}_\eta$ and multiplication with "scalars" in $F^+_{H_1}$.*
(2) *Given $0 \neq K \in F$, there is an automorphism f of $\mathcal{S}_\eta$ which is an isometry from H_1 to H_1K if and only if there are $X \in F^+_{H_1}$ and $Y \in F^-_{H_1}$ such that $K = X^2 - Y^2$.*

Proof. In view of Proposition 8.6.7 and Propositions 8.6.11, we can apply Lemma 8.6.15 to the underlying endomorphism (U, η). It follows then from Proposition 8.4.10 and Proposition 8.4.17 that we can transfer the uniqueness statement from Lemma 8.6.15 to the corresponding statement in part (1) above for compatible forms for $\mathcal{S}_\eta$.

We turn to proving part 2. If $K = X^2 - Y^2$ for some $X \in F^+_{H_1}$ and $Y \in F^-_{H_1}$, then an isometry is given by $(X + Y)\mathrm{Id}$ as in the proof of Lemma 8.6.16. So assume that H_1 is equivalent (isometric) to H_1K.

Without loss of generality we can assume that H_1 is given by one of the canonical compatible forms defined in Theorem 8.6.9. Indeed, suppose that the statement to be proved were true for those canonical forms (and let H_1 represent, for a moment, a form which is not necessarily such a canonical one). Suppose moreover that there exists an isometry f between H_1 and H_1K. Choose the appropriate canonical compatible form H_0 such that $\varepsilon(H_0) = \varepsilon(H_1) = \varepsilon(H_1K)$. Then, by part 2. above, there exists some $C \in F^+_{H_1}$ and an isometry $g : H_0 \to H_1C$. This will also be an isometry $g : H_0K \to H_1CK = H_1KC$. From all this we obtain the isometry

$$g^{-1}fg : H_0 \to H_0K.$$

By assumption, this implies that $K = X^2 - Y^2$ for some $X \in F^+_{H_1}$ and $Y \in F^-_{H_1}$.

So we can assume H_1 is a canonical compatible form. Now we wish to proceed in an analogous manner as we did in proving parts 1 and 2, but this time using Lemma 8.6.16. It remains only to check that the hypotheses of Lemma 8.6.16 are satisfied by (U, η).

Note that from Theorem 8.6.9 and Proposition 8.4.10 it follows that the coordinate matrix $H_1|_U$ is of the form

$$\begin{pmatrix} \vdots & \vdots & & \\ O & O & T & \\ O & -T & O & \cdots \\ T & O & O & \cdots \end{pmatrix} \in \mathbf{k}^{ml\times ml},$$

where $T \in \mathbf{k}^{l\times l}$ is chosen as in the proof of Theorem 8.6.9. In particular, H_1 is zero above the $l \times l$ block anti-diagonal, as required by Lemma 8.6.16. □

Part 3

Duality involutions and Morita theory

We transition now to the context of representations of (associative) $\mathbf{k}$-algebras. Given such an algebra A, for us a representation on a $\mathbf{k}$-vector space V is a map of algebras $A \longrightarrow \mathrm{End}(V)$ given by a right A-action

$$V \times A \longrightarrow V, \ (v, a) \longmapsto v \cdot a.$$

Thus representations of A are also called A-modules. It is well-known that the category of (right) modules Mod_A over an algebra A provides important information about A itself. Indeed, much of modern algebra is concerned with the study of categories of representations and their structures. A classical notion of equivalence between algebras is *Morita equivalence*, introduced in [**Mor58**]: two algebras A and B are Morita equivalent if their categories of right modules Mod_A and Mod_B are equivalent. An elegant reformulation of Morita equivalence between algebras can be obtained via the language of bicategories. Briefly, by regarding algebras as objects of a *Morita bicategory*[16], Morita equivalence corresponds to the notion of equivalence internal to a bicategory. Since the Morita bicategory is a convenient (higher) categorical environment where algebras and their equivalences live, it is natural to investigate the various structures that such a bicategory supports.

In this part of the thesis, which is essentially a reproduction of the paper [**LV19**], joint with A. Valentino, the protagonist is the notion of weak duality involution for bicategories. This concept was recently introduced in [**Shu18**] and is a bicategorical version of the notion of duality involution studied in Part 1. An archetypical example, defined on the bicategory of categories, is the operation of "taking the opposite category", together with opposite functors and natural transformations. In our work here, we study duality involutions in the context of Morita bicategories of algebras.

Concretely, we construct a canonical duality involution on the *fully dualisable* subbicategory of the Morita bicategory Alg_2 of *finite-dimensional* algebras. The full dualisability condition, the details of which we explain later, can be morally regarded as a finiteness condition on objects and 1-morphisms of a bicategory. The appearence of fully dualisable bicategories opens an interesting relation to the study of framed fully extended 2d topological quantum field theories, as in [**Lur17, Sch11**]. More precisely, the *core* of the fully dualisable part of Alg_2 corresponds to the symmetric monoidal bifunctors from the framed two-dimensional bordism category Bord_2^{fr} to Alg_2 itself. It is then natural to expect that Bord_2^{fr} comes equipped with a duality involution of geometric origin. Though this is one of the hidden motivations behind our work, we leave this line of research to future developments.

After quickly discussing how the 2-category $\mathrm{KV}_{\mathbf{k}}$ of *Kapranov-Voevodsky* vector spaces corresponds to the fully dualisable part of $\mathrm{LinCat}_{\mathbf{k}}$, we show that $\mathrm{KV}_{\mathbf{k}}$ can be canonically equipped with a *strict* duality involution. We then consider the bifunctor Rep which sends an algebra to its category of representation. We prove that Rep can be canonically equipped with all the necessary data of a *duality pseudofunctor*. Since Rep is an equivalence of bicategories, this can be regarded as an instance of the strictification theorem proven in [**Shu18**], which states that any bicategory with weak duality involution is biequivalent to a 2-category with strict duality involution via a duality pseudofunctor.

The constructions presented in Section 10.4 and 10.5 are structural enough to allow for a generalisation. In the last chapter of this part of the thesis we consider the case of

[16]See the discussion of the name at

`https://mathoverflow.net/questions/225701/reference-request-morita-bicategory` .

algebras in a symmetric semisimple finite tensor category $\mathcal{C}$, and their Morita bicategory $\mathrm{Alg}_2(\mathcal{C})$. We identify the target of the representation bifunctor $\mathrm{Rep}^{\mathcal{C}}$ as the 2-category $\mathrm{Mod}^{ss}(\mathcal{C})$ of semisimple *module categories* over $\mathcal{C}$. After equipping $\mathrm{Mod}^{ss}(\mathcal{C})$ with a weak duality involution, we argue that $\mathrm{Rep}^{\mathcal{C}}$ can be made into a duality pseudofuntor.

The material is organized as follows.

In Section 9.1 we review weak duality involutions on bicategories and duality pseudofunctors.

In Section 10.1 we provide some background material concerning modules over finite-dimensional algebras.

In Section 10.2 we briefly recall some basic aspects of the Morita bicategory of finite dimensional algebras, and we fix some notation regarding modules over an algebra.

In Section 10.3 we discuss finite linear categories and illustrate some properties of fully dualisable bicategories. We also discuss Kapranov-Voevodsky vector spaces and the representation bifunctor.

In Section 10.4 we construct a weak duality involution on the fully dualisable subbicategory Alg_2^{fd} of the Morita bicategory of finite dimensional algebras. This is the content of Theorem 10.4.2.

In Section 10.5, in Theorem 10.5.3, we show that the representation bifunctor $\mathrm{Rep} : \mathrm{Alg}_2^{fd} \to \mathrm{KV}_{\mathbf{k}}$ can be canonically equipped with the structure of a duality pseudofunctor, providing a strictification biequivalence.

Finally, in Chapter 11 we briefly describe a generalisation of the results obtained in the previous sections. In particular, we consider module categories and argue that they come equipped with a canonical weak duality involution. We then state a claim concerning the representation pseudofunctor $\mathrm{Rep}^{\mathcal{C}} : \mathrm{Alg}_2^{fd}(\mathcal{C}) \to \mathrm{Mod}^{ss}(\mathcal{C})$.

Throughout this part of the thesis we assume the reader to be familiar with the language of bicategories and associated higher categorical constructions. Also, we always assume the field $\mathbf{k}$ has characteristic 0 and is algebraically closed.

CHAPTER 9

Duality involutions on bicategories

9.1. Definition and basic theory

In this section we briefly recall the notion of a duality involution on a bicategory as introduced in [**Shu18**], which we also use as the main source for the details needed in the present section.

In the following, $\mathcal{A}$ and $\mathcal{B}$ denote bicategories.

Definition 9.1.1. *Let $\mathcal{A}$ be a bicategory. Then $\mathcal{A}^{co}$ denotes the bicategory with the same objects as $\mathcal{A}$, and*

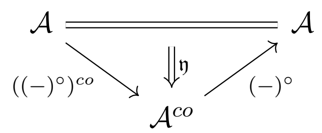
$$\mathcal{A}^{co}(x,y) := \mathcal{A}(x,y)^{op}, \quad \forall x,y \in \mathcal{A}. \tag{291}$$

In other words, $\mathcal{A}^{co}$ is the bicategory obtained from $\mathcal{A}$ by reversing 2-morphisms.[1]

One has that any bifunctor $F : \mathcal{A} \to \mathcal{B}$ induces a bifunctor $F^{co} : \mathcal{A}^{co} \to \mathcal{B}^{co}$, defined in the obvious way, and similarly for natural transformations and their modifications[2].

Definition 9.1.2. *A* weak duality involution *on $\mathcal{A}$ is defined via the following collection of data:*

- *a pseudofunctor $(-)^\circ : \mathcal{A}^{co} \to \mathcal{A}$;*
- *a pseudonatural adjoint equivalence $(\mathfrak{y}, \mathfrak{y}^\Box, \alpha, \alpha^\Box)$ in $[\mathcal{A},\mathcal{A}]$, with*



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In particular, $\mathfrak{y}$ consists of 1-cells

$$\mathfrak{y}_x : x \to x^{\circ\circ} \quad \forall x \in \mathcal{A} \tag{292}$$

and invertible 2-cells

$$\mathfrak{y}_f : \mathfrak{y}_y f \Longrightarrow f^{\circ\circ}\mathfrak{y}_x \qquad \forall f : x \to y \in \mathcal{A}; \tag{293}$$

- *an invertible modification $\zeta : \eta \star 1_{(-)^\circ} \longrightarrow 1_{(-)^\circ} \star \eta^{co}$, given in components by invertible 2-cells*

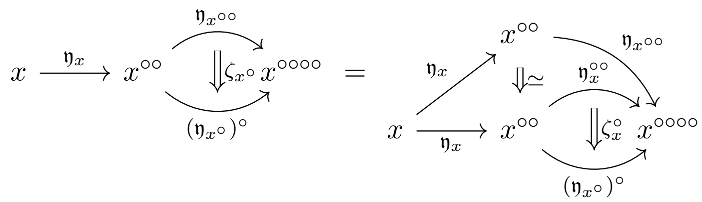
$$\zeta_x : \mathfrak{y}_{x^\circ} \Longrightarrow (\mathfrak{y}_x)^\circ, \quad \forall x \in \mathcal{A}. \tag{294}$$

[1] The "co" is not an abbreviation (such as "op" is an abbreviation for "opposite"); rather it is used in the sense of a prefix which often means something like "complementary". In the context of category theory, the prefix "co" is more specticifcally often employed when a certain directionality is reversed; compare with "product and co-product" or "algebra and co-algebra".

[2] Beware of the fact that $\theta^{co} : \gamma^{co} \to \eta^{co}$ for a modification $\theta : \eta \to \gamma$.

This data is required to satisfy, for every $x \in \mathcal{A}$, the following coherence condition

(295)



where, in the right-hand diagram, the unlabeled 2-cell is the coherence 2-cell $\mathfrak{y}_f$ from (300) for the special case when $f = \mathfrak{y}_x$.

If $(-)^\circ$ is a strict[3] pseudofunctor, $\mathfrak{y}$ is a strict pseudonatural isomorphism, and ζ is the identity modification, we speak of a *strong duality involution* on $\mathcal{A}$. Moreover, if in the case before $\mathfrak{y}$ is the identity as well, we have a *strict duality involution* on $\mathcal{A}$.

EXAMPLE 9.1.3. A prototypical example of a (strict) duality involution is provided by taking the opposite category. Indeed, denote with Cat the 2-category of small categories, and consider the following 2-functor

(296) $$(-)^{op} : \mathrm{Cat}^{co} \to \mathrm{Cat}$$

defined as follows:

- to a category $\mathcal{C}$ it assigns the opposite category $\mathcal{C}^{op}$;
- to a functor F between $\mathcal{C}$ and $\mathcal{D}$ it assigns the opposite functor F^{op} between $\mathcal{C}^{op}$ and $\mathcal{D}^{op}$; and
- to a natural transformation ϵ between F and G it assigns the opposite natural transformation ϵ^{op} between G^{op} and F^{op}.

Note that $(-)^{op}$ is defined on Cat^{co} since taking the opposite of a natural transformation between functors changes its direction.

Since taking the opposite twice is strictly the identity operation, we can choose the components of $\mathfrak{y}$ to be the identity 1-cells; moreover, we can choose the 2-cells witnessing the naturality to be identity 2-cells as well. Finally, if we choose the components of ζ to be identity 2-cells also, one can easily show that the above data satisfy the required compatibility diagram. Hence, we have that $(-)^{op}$ canonically provides a strict duality involution on Cat.

EXAMPLE 9.1.4. Consider the bicategory LinRel where 0-cells are finite-dimensional vector spaces over a fixed ground field $\mathbf{k}$, 1-cells are linear relations, and 2-cells are inclusions between linear relations. In other words, given vector spaces V and W, the "hom-category" $\mathsf{LinRel}(V, W)$ is the poset (with respect to inclusion) of linear subspaces of $V \oplus W$, viewed as a category.

We proceed to define a duality involution on LinRel. First, we define a pseudofunctor $(-)^\circ : \mathsf{LinRel}^{co} \to \mathsf{LinRel}$ as follows:

- given a 0-cell V, we define $V^\circ := V^* = \mathrm{Hom}(V, \mathbf{k})$;
- given a 1-cell $R : V \to W$, we define $R^\circ := (R^*)^\dagger$ as in Example 4.2.17, i.e.

$$(R^*)^\dagger = \{(\xi, \chi) \in V^* \oplus W^* \mid \xi(v) = \chi(w)\ \forall (v, w) \in R\};$$

- given a 2-cell $R \subseteq Q$, "$\subseteq^\circ$" is the inclusion $R^\circ \supseteq Q^\circ$.

[3] This requires $\mathcal{A}$ to be a strict bicategory.

It is straightforward to check that "$(-)^\circ$" is indeed a pseudofunctor. The associator coherence isomorphisms are equalities (as they must be, since equalities are the only invertible inclusion relations); indeed, the composition of linear relations is associative on the nose. The coherence isomorphisms for identity 1-cells are also equalities, i.e. $1_{V^\circ} = (1_V)^\circ$.

Next we define the pseudonatural transformation $\mathfrak{y} : 1_{\mathsf{LinRel}} \Rightarrow (-)^\circ \circ ((-)^\circ)^{co}$ for the duality involution. For this we need to define, for all 0-cells V, a 1-cell $\mathfrak{y}_V : V \to V^{\circ\circ}$, and for all 1-cells R, an invertible 2-cell $\mathfrak{y}_R : \mathfrak{y}_W R \Rightarrow R^{\circ\circ}\mathfrak{y}_V$. We let $\mathfrak{y}_V : V \to V^{**}, v \mapsto \mathrm{ev}_v$ be the standard canonical isomorphism. And for the $\mathfrak{y}_R$ we have equalities: indeed, given $R : V \to W$, we have, on the one hand,

$$\mathfrak{y}_W R = \{(v, L) \in V \oplus W^{**} \mid \exists w \in W : L = \mathrm{ev}_w \text{ and } (w, v) \in R\} \tag{297}$$

while on the other hand

$$R^{\circ\circ}\mathfrak{y}_V = \{(v, L) \in V \oplus W^{**} \mid (\mathrm{ev}_v, L) \in R^{\circ\circ}\}. \tag{298}$$

To see that $\mathfrak{y}_W R \subseteq R^{\circ\circ}\mathfrak{y}_V$, let $(v, L) \in \mathfrak{y}_W R$, i.e. $L = \mathrm{ev}_w$ for some $w \in W$ such that $(w, v) \in R$. Then $(\mathrm{ev}_v, \mathrm{ev}_w) \in R^{\circ\circ}$, because, for any $(\xi, \zeta) \in R^\circ$,

$$\mathrm{ev}_w(\xi) = \xi(w) = \zeta(v) = \mathrm{ev}_v(\zeta),$$

where the middle equality holds since $(v, w) \in R$. The inclusion $\mathfrak{y}_W R \subseteq R^{\circ\circ}\mathfrak{y}_V$ is in fact an equality, since $\dim R = \dim R^{\circ\circ}$, c.f. Remark 2.2.10.

In order to exhibit $\mathfrak{y}$ as part of an adjoint equivalence $(\mathfrak{y}, \mathfrak{y}^\square, \alpha, \alpha^\square)$, we let $\mathfrak{y}^\square$ be defined by components which are the inverses of the corresponding components of $\mathfrak{y}$. It is easily seen that, since all the 2-cells involved are idenities, $\mathfrak{y}^\square \circ \mathfrak{y}$ is the identity transformation; thus we may choose the unit α of the adjunction to also be the identity. By Theorem 1.4.7, this determines an adjoint equivalence $(\mathfrak{y}, \mathfrak{y}^\square, \alpha, \alpha^\square)$.

As a last piece of data, we need to specify the invertible modification ζ involved in the definition of a duality involution, namely we need the equality $\mathfrak{y}_{V^\circ} = (\mathfrak{y}_V)^\circ$ for every V. These equalities are indeed given: in Example 4.2.17 it was shown that $\mathfrak{y}_{V^*}$ and $(\mathfrak{y}_V^*)^\dagger$ are equal as linear relations $V^* \to V^{***}$.

Finally, we must check the coherence condition (295). But this is trivial, since all 2-cells involved are equalities. △

Definition 9.1.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be bicategories equipped with weak duality involutions, both denoted $(-)^\circ$. A* duality pseudofunctor *from $\mathcal{A}$ to $\mathcal{B}$ is a pseudofunctor $F : \mathcal{A} \to \mathcal{B}$ equipped with*

- *a pseudonatural adjoint equivalence $(\mathfrak{i}, \mathfrak{i}^\square, \alpha, \alpha^\square)$ in $[\mathcal{A}^{co}, \mathcal{B}]$, with*

$$\begin{array}{ccc} \mathcal{A}^{co} & \xrightarrow{F^{co}} & \mathcal{B}^{co} \\ {\scriptstyle (-)^\circ}\downarrow & \swarrow_{\mathfrak{i}} & \downarrow{\scriptstyle (-)^\circ} \\ \mathcal{A} & \xrightarrow[F]{} & \mathcal{B}. \end{array}$$

In particular, $\mathfrak{i}$ is specified by 1-cells

$$\mathfrak{i}_x : (F^{co}x)^\circ \longrightarrow F(x^\circ) \quad \forall x \in \mathcal{A}^{co} \tag{299}$$

and invertible 2-cells

$$\mathfrak{i}_f : \mathfrak{i}_y(F^{co}(f))^\circ \Longrightarrow F(f^\circ)\mathfrak{i}_x \qquad \forall f : x \to y \in \mathcal{A}^{co}; \tag{300}$$

- *an invertible modification θ whose components are 2-morphisms in $\mathcal{B}$ of the following form*

$$\begin{array}{ccc} (Fx)^{\circ\circ} & \xrightarrow{(\mathfrak{i}_x)^\circ} & (F(x^\circ))^\circ \\ {\scriptstyle \mathfrak{y}_{Fx}}\big\uparrow & \Downarrow \theta_x & \big\downarrow {\scriptstyle \mathfrak{i}_{x^\circ}} \\ Fx & \xrightarrow[F(\mathfrak{y}_x)]{} & F(x^{\circ\circ}) \end{array} \quad , \forall x \in \mathcal{A},$$

  satisfying a compatibility diagram involving ζ, $\mathfrak{y}$ and $\mathfrak{i}$; see [**Shu18**].

Similar to the case of a weak duality involution, we have the notion of a *strong duality pseudofunctor* and *strict duality pseudofunctor*.

The notion of a duality pseudofunctor allows to formulate the following theorem, which is one of the main results in [**Shu18**].

THEOREM 9.1.6. *Let $\mathcal{A}$ be a bicategory with a weak duality involution. Then there exists a 2-category $\mathcal{A}'$ with a strict duality involution and a duality pseudofunctor $\mathcal{A} \to \mathcal{A}'$ that is a biequivalence.*

The theorem above is essentially a coherence theorem for bicategories with duality involutions, which ensures that there is no loss in generality in considering only strict duality involutions. In [**Shu18**], the theorem is proven by using the theory of 2-monads and representable multicategories.

In Theorem 10.5.3, which constitutes the main result of the next chapter, we provide a concrete illustration of the above theorem involving naturally occurring bicategories with duality involutions.

CHAPTER 10

Bicategories of algebras and duality

10.1. Modules over finite-dimensional algebras

In the following, we recall the basic material that we need regarding finite-dimensional modules over finite-dimensional **k**-algebras. We fix a field **k** which is of characteristic 0 and algebraically closed. We will mainly follow [**SY11**], to which we refer the reader for the proofs of the various statements.

Let A be a finite-dimensional **k**-algebra. Recall that the category Mod_A of finite-dimensional right modules over A is an abelian category. Recall also that for any right A-module M, the vector space $\hom_A(M, A)$ comes equipped canonically with a left A-module structure induced by left multiplication on A.

DEFINITION 10.1.1. *An object $P \in \mathrm{Mod}_A$ is called* projective *if the functor* $\hom_A(P, -) : \mathrm{Mod}_A \to \mathrm{Vect}_{\mathbf{k}}$ *is exact.*

THEOREM 10.1.2. *Let A be a* semisimple *finite-dimensional* **k**-*algebra. Then any finite-dimensional module over A is projective.*

We recall also the notion of tensor product over an algebra

DEFINITION 10.1.3. *Let M and N be a right and left A-module, respectively. The* tensor product over A *of M and N is the vector space given by*

$$M \otimes_A N := \{x \otimes y \mid x \in M, y \in N\} \, / \, \{xa \otimes y - x \otimes ay\} \tag{301}$$

The following lemma is immediate

LEMMA 10.1.4. *Let M and N be a right and left A-module, respectively. The canonical braiding on* $\mathrm{Vect}_{\mathbf{k}}$ *induces a linear isomorphism*

$$M \otimes_A N \simeq N \otimes_{A^{op}} M \tag{302}$$

Notice that if M is a (B, A)-bimodule and N is a (A, C)-bimodule, then $M \otimes_A N$ canonically inherits a (B, C)-bimodule structure. Moreover, the isomorphism in Lemma 10.1.4 is compatible with this bimodule structure.

The following theorem, called the *adjoint theorem*, asserts that for any (A, B)-bimodule, the functors $(-) \otimes_A M$ and $\hom_B(M, -)$ form an adjoint pair

THEOREM 10.1.5. *Let A and B be* **k**-*algebras, and let M be a (A, B)-bimodule. Then for any right A-module X and right B-module Y the linear map*

$$\begin{aligned} \hom_B(X \otimes_A M, Y) &\longrightarrow \hom_A(X, \hom_B(M, Y)) \\ g &\longmapsto (x \longmapsto f_x : m \longmapsto g(x \otimes m)) \end{aligned} \tag{303}$$

is an isomorphism.

In the case in which Y is a (C, B)-bimodule, the vector spaces $\hom_B(X \otimes_A M, Y)$ and $\hom_A(X, \hom_B(M, Y))$ aquire a canonical structure of left C-module, induced by the left

C-action on Y. It is routine to show that the isomorphism in Theorem 10.1.5 is C-linear. Similarly if X is a (C, A)-module.

THEOREM 10.1.6. *Let A be a $\mathbf{k}$-algebra, and let P be a projective right A-module. Then:*

- $\hom_A(P, A)$ *is a projective left A-module; and*
- *the linear map*

$$\begin{aligned} \psi_P : P &\to \hom_{A^{op}}(\hom_A(P, A), A^{op}) \\ p &\longmapsto (\psi_P(p) : g \longmapsto g(p)) \end{aligned} \tag{304}$$

is an isomorphism of right A-modules.

Notice that in the above theorem, we regard a left (right) A-module as a right (left) A^{op}-module. Similar to the previous theorem, in the case in which P is a (B, A)-bimodule, it is routine to check that the isomorphism in Theorem 10.1.6 is B-linear. Moreover, ψ_P is natural in P.

THEOREM 10.1.7. *Let A and B be $\mathbf{k}$-algebras, and let P be a (B, A)-bimodule which is projective as a right A-module. Then for any right A-module X the linear map*

$$\begin{aligned} X \otimes_A \hom_A(P, A) &\to \hom_A(P, X) \\ x \otimes g &\longmapsto (p \longmapsto x \cdot g(p)) \end{aligned} \tag{305}$$

is an isomorphism of right B-modules and natural in X.

Again, if X is a (C, A)-bimodule, it is routine to check that the isomorphism in Theorem 10.1.7 is C-linear.

10.2. The Morita bicategory of algebras

In this section we briefly review some aspects of the Morita bicategory of algebras, and some standard notation regarding modules and bimodule over finite-dimensional algebras. We refer the reader to [**Ben67, Lei98**] for the terminology and details concerning bicategories and their functors.

DEFINITION 10.2.1. *The* Morita bicategory of algebras Alg_2 *is the bicategory where*

- *the objects are finite-dimensional $\mathbf{k}$-algebras;*
- *the 1-morphisms from A to B are finite-dimensional $\mathbf{k}$-vector spaces which are (A, B)-bimodules; and*
- *the 2-morphisms are intertwiners of bimodules, i.e. $\mathbf{k}$-linear maps of bimodules which are compatible with the respective left and right actions of $\mathbf{k}$-algebras.*

Throughout the chapter, the terms “algebra” and “bimodule” will always refer to the sort appearing in the definition of Alg_2; similarly for “right modules”, etc.. Moreover, if M is an (A, B)-bimodule, we indicate this by writing ${}_AM_B$, though at times we will drop the subscripts. In the following, we schematically recall some basic features of Alg_2 and related notions which will be useful in later sections of the chapter.

The composition operations for 1- and 2-morphisms in Alg_2 are defined as follows[1]

- *composition of 1-morphisms*: for ${}_AM_B$ and ${}_BN_C$ bimodules, their composition is defined as

$$N \circ M := {}_A(M \otimes_B N)_C \tag{306}$$

[1] We work under the assumption that representatives for tensor products have been fixed.

- *horizontal composition of 2-morphisms*: for $f : {}_AM_B \to {}_AN_B$ and $g : {}_BM'_C \to {}_BN'_C$ intertwiners of bimodules, their horizontal composition is defined as

$$g \bullet^h f := f \otimes_B g \tag{307}$$

- *vertical composition of 2-morphisms*: for $f : {}_AM_B \to {}_AN_B$ and $g : {}_AN_B \to {}_AP_B$ intertwiners, their vertical composition is defined as

$$g \bullet^v f := g \circ f \tag{308}$$

The coherence data for the bicategory Alg_2 arise as follows

- *associators*: for ${}_AM_B$, ${}_BN_C$ and ${}_CP_D$, the associator isomorphism

$$\alpha_{M,N,P} : P \circ (N \circ M) \xrightarrow{\simeq} (P \circ N) \circ M \tag{309}$$

is given by the canonical isomorphism $(M \otimes_B N) \otimes_C P \simeq M \otimes_B (N \otimes_C P)$ of tensor products of bimodules;
- *unitors*: for any algebra A, the unit 1-morphism $1_A : A \to A$ is given by A itself regarded as a (A, A)-bimodule; for any (A, B)-bimodule M, the left and right unitor isomorphisms

$$1_B \circ M \simeq M \simeq M \circ 1_A \tag{310}$$

are given by the canonical isomorphisms $M \otimes_B B \simeq M \simeq A \otimes_A M$.

The isomorphisms above satisfy the compatibility diagrams for the coherence data of a bicategory.

REMARK 10.2.2. Our notation for the Morita bicategory of algebras differs from the one used in [**Lur09**].

Recall the following

DEFINITION 10.2.3. *Let $\mathcal{B}$ be a bicategory. A 1-morphism $f : x \to y$ is called an* equivalence *if there exists a 1-morphism $g : y \to x$, and invertible 2-morphisms*

$$id_x \simeq g \circ f, \quad f \circ g \simeq id_y \tag{311}$$

Two objects x and y in a bicategory $\mathcal{B}$ are called *equivalent* if there exists an equivalence between x and y.

The following is a well-known result.

PROPOSITION 10.2.4. *Two algebras are equivalent as objects in* Alg_2 *if and only if they are Morita equivalent.*

The following notation will be used (hopefully) consistently throughout the chapter.

If ${}_AM_B$ and ${}_CM'_D$ are bimodules, we denote with $\hom_{\mathbf{k}}(M, M')$ the vector space of $\mathbf{k}$-linear maps from M to M'. Given bimodules ${}_AM_B$ and ${}_DN_B$, $\hom_B(M, N)$ denotes the set of right B-module morphisms, namely elements of $\hom_{\mathbf{k}}(M, N)$ which, additionally, are compatible with the right B-action. We avoid completely the analogous notion for left modules, so that our notation for morphisms of right-modules is unambiguous.

Given bimodules ${}_AM_B$ and ${}_CN_B$, the vector space $\hom_B(M, N)$ may naturally be equipped with a left C-action and a right A-action. Indeed, these are defined as

$$\begin{aligned} \hom_B(M, N) \times A &\to \hom_B(M, N) \\ (f, a) &\mapsto fa : x \mapsto f(ax) \end{aligned} \tag{312}$$

and

$$\begin{aligned} C \times \hom_B(M,N) &\mapsto \hom_B(M,N) \\ (c,f) &\mapsto cf : x \mapsto cf(x), \end{aligned} \tag{313}$$

respectively. We write ${}_C\hom_B(M,N)_A$ to indicate this bimodule structure, and we always assume these left and right actions unless otherwise indicated.

Recall that from any algebra A we obtain an opposite algebra A^{op}. This is the $\mathbf{k}$-algebra which has the same underlying vector space as A, and the same unit, but where the multiplication is inverted. For notational ease, we denote multiplication using juxtaposition when it is clear which algebra A is at play; the notation $\star$ indicates when we are multiplying in the opposite algebra.

Finally, recall that any left A-module can be viewed as right A^{op}-module. Indeed, for M a left A-module we can consider the following right A^{op}-action

$$\begin{aligned} M \times A^{op} &\to M \\ (m,a) &\mapsto a \cdot m \end{aligned} \tag{314}$$

It is readily checked that this does indeed define a right action. In a similar fashion, bimodules ${}_AM_B$ may be viewed as bimodules ${}_{B^{op}}M_{A^{op}}$. We will make this kind of switch tacitly when no confusion is to be feared.

10.3. Finite linear categories and dualisability

In this section we provide a review of well-known material, mainly following [**EGNO15, EO04, DSS19**]. This will be useful both to give a precise characterisation of the bifunctor Rep, and in view of the general setting of Chapter 11.

10.3.1. Finite linear categories. For $\mathbf{k}$ a fixed ground field, recall that a *linear category* is an abelian category enriched over $\mathrm{Vect}_{\mathbf{k}}$, the symmetric monoidal category of $\mathbf{k}$-vector spaces, not necessarily finite-dimensional. A *linear functor* is an additive functor which is also a functor of $\mathrm{Vect}_{\mathbf{k}}$-enriched categories.

Definition 10.3.1. *A linear category $\mathcal{C}$ is called* finite *if:*

- *$\mathcal{C}$ has finite-dimensional vector spaces as spaces of morphisms;*
- *every object of $\mathcal{C}$ has finite length;*
- *$\mathcal{C}$ has enough projectives; and*
- *there are finitely many isomorphism classes of simple objects.*

Example 10.3.2. An example of a finite linear category is $\mathsf{Vect}_{\mathbf{k}}$, the category of finite-dimensional $\mathbf{k}$-vector spaces.

The following proposition is important for recognizing finite linear categories.

Proposition 10.3.3. *A linear category $\mathcal{C}$ is finite if and only if it is equivalent to the category* Mod_A *of finite-dimensional (right) modules over a finite-dimensional* $\mathbf{k}$*-algebra* A*.*

Recall that an additive functor between abelian categories is called *left exact* if it sends left exact sequences to left exact sequences. The notion of a right exact functor is similar.

We can consider now the 2-category of finite linear categories.

Definition 10.3.4. *For a fixed ground field* $\mathbf{k}$*, the 2-category* $\mathrm{LinCat}_{\mathbf{k}}$ *has:*

- *finite linear categories as objects;*

- *right exact functors as 1-morphisms; and*
- *natural transformations as 2-morphisms.*

We can consider $\mathrm{LinCat}_{\mathbf{k}}$ as a linearization of Alg_2 via the *representation bifunctor*. More precisely, consider the bifunctor

$$\mathrm{Rep} : \mathrm{Alg}_2 \to \mathrm{LinCat}_{\mathbf{k}} \tag{315}$$

defined as follows:

- to a finite-dimensional algebra A it assigns the category Mod_A;
- to a finite-dimensional (A, B)-bimodule ${}_AM_B$ it assigns the right exact functor $(-) \otimes_A M_B : \mathrm{Mod}_A \to \mathrm{Mod}_B$; and
- to an intertwiner f between ${}_AM_B$ and ${}_AN_B$ it assigns the corresponding natural transformation between $(-) \otimes_A M_B$ and $(-) \otimes_A N_B$.

Notice that the various isomorphisms needed to make Rep into a bifunctor arise canonically from the properties of the tensor product of modules.

As pointed out in [**BDSV15**], following [**DSS19, Sch11**] one obtains

Proposition 10.3.5. *The bifunctor* Rep *is an equivalence of bicategories*

Remark 10.3.6. The bifunctor Rep is actually an equivalence of symmetric monoidal bicategories. See [**Sta16**] and [**Sch11**] for details on symmetric monoidal structures on bicategories.

10.3.2. Full dualisability. We now recall some basic notions concerning adjoints for 1-morphisms in bicategories and full-dualisability.

Let $\mathcal{B}$ be a bicategory.

Definition 10.3.7. *A 1-morphism $f : x \to y$ in $\mathcal{B}$* admits a right adjoint *if there exists a 1-morphism $g : y \to x$, and 2-morphisms $\epsilon : f \circ g \to id_y$ and $\eta : id_x \to g \circ f$ satisfying the triangle identities.*

Similarly, we have the notion of a left adjoint of a 1-morphism.

An *adjunction* $f \dashv g$ is a collection (f, g, ϵ, η) such that g is a right adjoint to f via ϵ and η. We say that $f \dashv g$ is an *adjoint equivalence* if ϵ and η are invertible 2-morphisms.

The following theorem will be useful in later sections.

Theorem 10.3.8 ([**Gur12**]). *Let $\mathcal{B}$ be a bicategory, and let f be an equivalence in $\mathcal{B}$. Then f is part of an adjoint equivalence $f \dashv g$.*

Remark 10.3.9. As remarked in [**Gur12**], the theorem above guarantees something stronger than the existence of an adjoint equivalence. Indeed, given an equivalence $f : x \to y$ in $\mathcal{B}$, a (pseudo) inverse g and a 2-isomorphism $\alpha : f \circ g \simeq \mathrm{id}_y$, there exists a *unique* adjoint equivalence (f, g, ϵ, η) with $\epsilon = \alpha$.

Example 10.3.10. Let $\mathcal{C}$ be a monoidal category. If we regard $\mathcal{C}$ as a bicategory with a single object, then a 1-morphism x admits a right (resp. left) adjoint if and only if x admits a left (resp. right) dual as an object in $\mathcal{C}$.

Definition 10.3.11. *A bicategory $\mathcal{B}$ is said to* admit duals for 1-morphisms *if any 1-morphism admits a right and a left adjoint.*

In the following we recall the definition of duals in symmetric monoidal bicategories; see [**Sch11**] for details.

DEFINITION 10.3.12. *Let* $(\mathcal{B}, \otimes, \mathbf{1})$ *be a symmetric monoidal bicategory. An object* $x \in \mathcal{B}$ *is* dualisable *if there exists* $x^* \in \mathcal{B}$ *and 1-morphisms* $e : x \otimes x^* \to \mathbf{1}$ *and* $c : \mathbf{1} \to x^* \otimes x$ *sastisfying the zig-zag identities up to 2-isomorphisms.*

REMARK 10.3.13. The statement regarding the zig-zag equations means that for any dualisable object $x \in \mathcal{B}$ there are isomorphisms

$$\begin{aligned} (e \otimes \mathrm{id}_x) \circ (\mathrm{id}_x \otimes c) &\simeq \mathrm{id}_x \\ (\mathrm{id}_{x^*} \otimes e) \circ (c \otimes \mathrm{id}_{x^*}) &\simeq \mathrm{id}_{x^*}. \end{aligned} \tag{316}$$

See for instance [**Lur09**].

DEFINITION 10.3.14. *A symmetric monoidal bicategory* $\mathcal{B}$ *is said to* admit duals for objects *if any object is dualisable.*

We can combine the two requests on a bicategory via the following

DEFINITION 10.3.15. *A symmetric monoidal bicategory* $\mathcal{B}$ *is said to be* fully dualisable *if it admits duals for objects and 1-morphisms.*

Given a symmetric monoidal bicategory $\mathcal{B}$, we denote with $\mathcal{B}^{fd}$ the maximal sub-bicategory of $\mathcal{B}$ which is fully dualisable. An object in $\mathcal{B}^{fd}$ is called *fully dualisable.*

We now discuss the fully dualisable part of the (symmetric monoidal[2]) bicategories of interest for the present work, namely Alg_2 and $\mathrm{LinCat}_{\mathbf{k}}$; our main reference will be Appendix A of [**BDSV15**].

From [**Dav11, Sch11**] it follows that Alg_2^{fd} corresponds to the full sub-bicategory of Alg_2 spanned by semi-simple[3] (finite-dimensional) $\mathbf{k}$-algebras. Note that any finite-dimensional module over a semi-simple finite-dimensional algebra is automatically projective; see Section 10.1.

To discuss the fully dualisable part of $\mathrm{LinCat}_{\mathbf{k}}$, we need first the following

DEFINITION 10.3.16. *A Kapranov-Voevodsky (KV) vector space is a finite linear category which is semi-simple and equivalent to* $\mathrm{Vect}_{\mathbf{k}}^n$ *for some* n.

From [**BDSV14**] we have that $\mathrm{LinCat}_{\mathbf{k}}^{fd}$ is the full[4] sub-2-category of $\mathrm{LinCat}_{\mathbf{k}}$ spanned by KV-vector spaces. For simplicity we use $\mathrm{KV}_{\mathbf{k}}$ to denote $\mathrm{LinCat}_{\mathbf{k}}^{fd}$.

From the fact that Rep is a symmetric monoidal biequivalence one has

PROPOSITION 10.3.17. *The bifunctor* Rep *induces by restriction an equivalence of bicategories between* Alg_2^{fd} *and* $\mathrm{KV}_{\mathbf{k}}$.

The proposition above is guaranteed by the fact that any symmetric monoidal bifunctor $\mathcal{A}^{fd} \to \mathcal{B}$ factors uniquely through $\mathcal{B}^{fd}$, and by the maximality property of fully dualisable subcategories.

REMARK 10.3.18. The definition of a Kapranov-Voevodsky vector space provided above is slightly different from that in [**KV94**]; see Chapter 11 for comments.

[2]We will not indulge in the gory details of their symmetric monoidal products.

[3]Semi-simplicity arises from the assumption that $\mathbf{k}$ has characteristic 0; *separability* is a suitable notion otherwise.

[4]Note that any right exact functor between semi-simple abelian categories is automatically left exact.

In the following, we assume $\mathrm{KV}_{\mathbf{k}}$ to be equipped with the strict duality involution induced by taking the opposite category. We note that this operation does *not* provide a strict duality involution on $\mathrm{LinCat}_{\mathbf{k}}$: the opposite category of a finite linear category is again a finite linear category, but the opposite of a right exact functor is *left* exact. However, morphisms between KV-vector spaces are exact functors, so $(-)^{op}$ is a strict duality involution on $\mathrm{KV}_{\mathbf{k}}$.

10.4. A duality involution on Alg_2^{fd}

In this section we explicitely construct a weak duality involution on the Morita bicategory Alg_2^{fd}. In the next section we will then prove that such a weak duality involution *strictifies* to the duality involution $(-)^{op}$ on $\mathrm{KV}_{\mathbf{k}}$.

Consider the pseudofunctor

$$(-)^\circ : (\mathrm{Alg}_2^{fd})^{co} \to \mathrm{Alg}_2^{fd} \tag{317}$$

defined as follows:

- to an object, i.e an algebra A it assigns $A^\circ := A^{op}$, the opposite algebra;
- to a 1-morphism, i.e. a bimodule ${}_AM_B$ it assigns $({}_AM_B)^\circ := {}_{A^{op}}(\mathrm{hom}_B(M,B))_{B^{op}}$[5]; and
- to a 2-morphism, i.e. an intertwiner f it assigns $f^\circ := f^*$.

In the above definition, f^* denotes the adjoint map, namely it is given by the operation "precompose with f".

For $(-)^\circ$ to be a pseudofunctor, we need to specify invertible 2-morphisms in Alg_2^{fd}

$$({}_AM \otimes_B N_C)^\circ \Rightarrow ({}_AM_B)^\circ \otimes_{B^{op}} ({}_BN_C)^\circ \tag{318}$$

and

$$1_A^\circ \Rightarrow 1_{A^\circ}. \tag{319}$$

satisfying compatibility diagrams.

First, notice that we have the following isomorphisms of (C,A)-bimodules

$$\begin{aligned}\mathrm{hom}_C(M \otimes_B N, C) &\simeq \mathrm{hom}_B(M, \mathrm{hom}_C(N,C)) \\ &\simeq \mathrm{hom}_C(N,C) \otimes_B \mathrm{hom}_B(M,B)\end{aligned} \tag{320}$$

which is natural in M and N; see Section 10.1. Notice now that for arbitrary bimodules ${}_AM_B$, ${}_BN_C$ and ${}_AS_C$, any morphism ${}_AM \otimes_B N_C \to {}_AS_C$ can be regarded as a morphism ${}_{C^{op}}N \otimes_{B^{op}} M_{A^{op}} \to {}_{C^{op}}S_{A^{op}}$. We then get the isomorphism in (318).

Consider now the natural isomorphism of *algebras*

$$\mathrm{hom}_A(A,A) \xrightarrow{\simeq} A^{op} \tag{321}$$

given by

$$f \longmapsto f(1). \tag{322}$$

It is straightforward to check that the above isomorphism is an isomorphism of (A,A)-bimodules, where we canonically regard A^{op} equipped with the $(A^{op})^{op} = A$ left and right actions. By regarding them both as (A^{op}, A^{op})-bimodules we obtain the isomorphism (319).

[5]Here we are taking the (B,A)-bimodule $\mathrm{hom}_B(M,B)$ and viewing it as an as (A^{op}, B^{op})-bimodule. As mentioned above, we will henceforth perform this operation tacitly without further remark.

Notice that the required naturality with respect to 2-morphisms of the isomorphism (318) and (319) is guaranteed by the naturality of the various isomorphisms of bimodules involved.

REMARK 10.4.1. The isomorphisms above, in particular (318), are available because the objects of Alg_2^{fd} are finite-dimensional *semisimple* algebras, and hence all bimodules are projective.

We now proceed to construct the pseudonatural adjoint equivalence $\mathfrak{y}$ and the modification ζ required in definition 9.1.2.

For $\mathfrak{y}$, which should be a pseudonatural transformation of bifunctors

$$1_{\mathrm{Alg}_2} \Rightarrow (-)^\circ \circ ((-)^\circ)^{co}, \tag{323}$$

we must define its component 1- and 2-morphisms, and exhibit $\mathfrak{y}$ as part of an adjoint equivalence $(\mathfrak{y}, \mathfrak{y}^\square, \alpha, \alpha^\square)$ in the functor bicategory $[\mathrm{Alg}_2, \mathrm{Alg}_2]$. Given A, we define the component 1-morphism $\mathfrak{y}_A : A \to ((A)^\circ)^\circ = A$ to be the identity bimodule ${}_AA_A$. Given a bimodule ${}_AM_B$, the component 2-morphism $\mathfrak{y}_M$ must be an invertible 2-cell which witnesses the "commutativity" of the square

$$\begin{array}{ccc} A & \xrightarrow{\mathfrak{y}_A} & A \\ {\scriptstyle M}\downarrow & \swarrow\!\!\!\!\swarrow_{\mathfrak{y}_M} & \downarrow{\scriptstyle M^{\circ\circ}} \\ B & \xrightarrow[\mathfrak{y}_B]{} & B. \end{array} \tag{324}$$

We define $\mathfrak{y}_M$ to be the isomorphism of bimodules (read left to right here) given by

$$ {}_A(A \otimes_A \hom_A(\hom_B(M, B), A))_B \simeq {}_A(\hom_A(\hom_B(M, B), A))_B \simeq {}_AM_B \simeq {}_A(M \otimes_B B)_B \tag{325}$$

where the middle isomorphism is the (inverse of the) canonical isomorphism ${}_AM_B \to {}_A(M^{\circ\circ})_B$; c.f. Theorem 10.1.6. It is straightforward to check that the components of $\mathfrak{y}$ satisfy the coherence axioms required by the definition of a transformation of bifunctors.

In order to exhibit $\mathfrak{y}$ as part of an adjoint equivalence, it is enough to specify an equivalence $(\mathfrak{y}, \mathfrak{y}^\square, \alpha, \alpha^\square)$, since then there exists a unique associated adjoint equivalance obtained by modifying the counit $\alpha^\square$ as necessary (see [**Gur12**]). We define the transformation

$$\mathfrak{y}^\square : (-)^\circ \circ ((-)^\circ)^{co} \Rightarrow 1_{\mathrm{Alg}_2} \tag{326}$$

by letting $\mathfrak{y}^\square_A : A^{\circ\circ} = A \to A$ be the identity bimodule ${}_AA_A$, and by letting

$$\mathfrak{y}^\square_M : {}_AM_B \circ \mathfrak{y}^\square_A \Rightarrow \mathfrak{y}^\square_B \circ ({}_AM_B)^{\circ\circ} \tag{327}$$

be the isomorphism of bimodules (read left to right) given by

$$ {}_A(A \otimes_A M)_B \simeq {}_AM_B \simeq {}_A(\hom_A(\hom_B(M, B), A))_B \simeq {}_A(\hom_A(\hom_B(M, B), A) \otimes_B B)_B. \tag{328}$$

For the unit α of the equivalence, which should be an invertible modification

$$\alpha : 1_{1_{\mathrm{Alg}_2}} \longrightarrow \mathfrak{y}^\square \circ \mathfrak{y}, \tag{329}$$

we define its component 2-cells to be the canonical isomorphisms

$$\alpha_A : (1_{1_{\mathrm{Alg}_2}})_A = {}_AA_A \Rightarrow {}_A(A \otimes_A A)_A = (\mathfrak{y}^\square \circ \mathfrak{y})_A. \tag{330}$$

We define the counit $\alpha^{\square} : \mathfrak{y} \circ \mathfrak{y}^{\square} \longrightarrow 1_{1_{\mathrm{Alg}_2}}$ analogously: we let its components be the isomorphisms

$$\alpha_A^{\square} : {}_A(A \otimes_A A)_A \Rightarrow {}_A A_A. \tag{331}$$

It is straightforward to check that α and $\alpha^{\square}$ do indeed defined modifications.

Now we define the invertible modification ζ, which means we need to specify invertible 2-cells

$$\zeta_A : \mathfrak{y}_{A^\circ} \Rightarrow (\mathfrak{y}_A)^\circ. \tag{332}$$

For fixed A, we choose as ζ_A the inverse of the isomorphism

$$(\mathfrak{y}_A)^\circ = {}_{A^{op}}(\mathrm{hom}_A(A, A))_{A^{op}} \to {}_{A^{op}}A^{op}{}_{A^{op}} = \mathfrak{y}_{A^\circ} \tag{333}$$

which already appeared as part of the coherence data for the bifunctor $(-)^\circ$, namely in (319). The coherence diagram from the definition of a modification reads, in this case, as

$$\begin{array}{ccc} {}_{A^{op}}(A^{op} \otimes_{A^{op}} M^{\circ\circ\circ})_{B^{op}} & \xRightarrow{1\star\zeta_A} & {}_{A^{op}}(\mathrm{hom}_A(A, A) \otimes_{A^{op}} M^{\circ\circ\circ})_{B^{op}} \\ {\scriptstyle (\eta\star(-)^\circ)_M}\Big\Downarrow & & \Big\Downarrow{\scriptstyle ((-)^\circ\star\eta^{co})_M} \\ {}_{A^{op}}(M^\circ \otimes_{B^{op}} B^{op})_{B^{op}} & \xRightarrow[\zeta_B\star 1]{} & {}_{A^{op}}(M^\circ \otimes_{B^{op}} \mathrm{hom}_B(B, B))_{B^{op}}. \end{array} \tag{334}$$

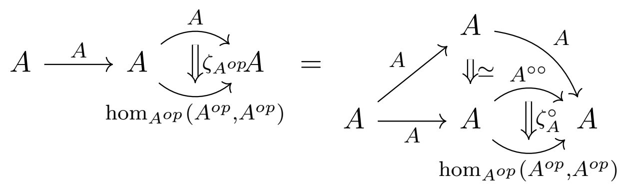
Verifying that this diagram commutes is straightforward yet tedious; we omit the details.

We can now state the following

THEOREM 10.4.2. *The bifunctor* $(-)^\circ$ *together with* $\mathfrak{y}$ *and* ζ *defines a weak duality involution on* Alg_2^{fd}.

PROOF. We need to verify that ζ satisfies the compatibility required for a weak duality involution as stated in [**Shu18**]. Namely, we need to show that for any $A \in \mathrm{Alg}_2^{fd}$ we have the following equality of 2-morphisms

(335)



First, recall that by construction

$$\begin{aligned} \zeta_{A^{op}} &: A \to \mathrm{hom}_{A^{op}}(A^{op}, A^{op}) \\ & x \longmapsto \phi_x : a \longmapsto a \cdot x \end{aligned} \tag{336}$$

and similarly

$$\begin{aligned} \zeta_A &: A^{op} \to \mathrm{hom}_A(A, A) \\ & x \longmapsto \bar{\phi}_x : a \longmapsto x \cdot a \end{aligned} \tag{337}$$

The LHS of (335) is then given by the isomorphism

$$\begin{aligned} \mathrm{id}_A \otimes \zeta_{A^{op}} &: A \otimes_A A \to A \otimes_A \mathrm{hom}_{A^{op}}(A^{op}, A^{op}) \\ & a \otimes b \longmapsto a \otimes \phi_b. \end{aligned} \tag{338}$$

REMARK 10.4.3. In the definitions above, the multiplication is *always* performed in A.

On the other hand, the RHS of (335) is the following composition

$$A \otimes_A A \xrightarrow{\mathfrak{y}_A} A \otimes_A A^{\circ\circ} \xrightarrow{\mathrm{id}_A \otimes (\zeta_A)^*} A \otimes_A \hom_{A^{op}}(A^{op}, A^{op}) \tag{339}$$

where $\mathfrak{y}_A$ is given by the 2-morphism defined in (328), namely it is given by the following composition

$$\begin{aligned} A \otimes_A A &\xrightarrow{\cong} A \xrightarrow{\cong} A^{\circ\circ} \xrightarrow{\cong} A \otimes_A A^{\circ\circ} \\ a \otimes b &\longmapsto a \cdot b \longmapsto f_{ab} \longmapsto 1 \otimes f_{ab} \end{aligned} \tag{340}$$

where

$$f_{ab}(g) := g(ab), \quad \forall g \in \hom_A(A, A). \tag{341}$$

Note now that $\forall x \in A^{op}$ we have the following

$$\begin{aligned} (\zeta_A)^*(f_{ab})(x) &= f_{ab}(\zeta_A(x)) \\ &= f_{ab}(\bar{\phi}_x) \\ &= \bar{\phi}_x(a \cdot b) \\ &= x \cdot a \cdot b \\ &= \phi_b(x \cdot a) = \phi_b(a \star x) \\ &= (\phi_b \star a)(x) \\ &= (a \cdot \phi_b)(x), \end{aligned} \tag{342}$$

where for clarity we use $\cdot$ to denote an A-action, and $\star$ to denote an A^{op}-action. Hence the isomorphism in (339) is explicitely given by

$$a \otimes b \longmapsto 1 \otimes a \cdot \phi_b = a \otimes \phi_b, \quad \forall a, b \in A \tag{343}$$

which agrees with the LHS in (335). □

REMARK 10.4.4. The weak duality involution $(-)^\circ$ on Alg_2^{fd} can be regarded as an instance of the procedure outlined in [**Shu18**, Ex. 2.10]. Our concrete description is needed in order to prove the main theorem in Section 10.5.

REMARK 10.4.5. We find it interesting to notice that the data needed to make $(-)^\circ$ into a duality involution is *entirely* produced from the coherence data needed to define Alg_2 and the pseudofunctor $(-)^\circ$ itself. A similar remark applies to the duality involution on $\mathrm{KV}_{\mathbf{k}}$, though the coherence data in this case is trivial.

10.5. Rep as a duality pseudofunctor

In this section we show that the bifunctor $\mathrm{Rep} : \mathrm{Alg}_2^{fd} \to \mathrm{KV}_{\mathbf{k}}$ introduced in Section 10.3 can be canonically equipped with the structure of a duality pseudofunctor.

According to Definition 9.1.5, we need to provide a pseudonatural adjoint equivalence $\mathfrak{i}$ and a modification θ satisfying a compatibility diagram.

Definition of $\mathfrak{i}$: we need to specify a pseudonatural equivalence of the following form

$$\begin{array}{ccc} (\mathrm{Alg}_2^{fd})^{co} & \xrightarrow{\mathrm{Rep}^{co}} & (\mathrm{KV}_{\mathbf{k}})^{co} \\ {\scriptstyle (-)^\circ}\downarrow & \overset{\mathfrak{i}}{\swarrow\!\!\swarrow} & \downarrow{\scriptstyle (-)^{op}} \\ \mathrm{Alg}_2^{fd} & \xrightarrow[\mathrm{Rep}]{} & \mathrm{KV}_{\mathbf{k}}. \end{array} \tag{344}$$

This consists of a family of invertible 1-morphisms in $\mathrm{KV}_{\mathbf{k}}$

$$\mathfrak{i}_A : \mathrm{Rep}(A)^{op} \longrightarrow \mathrm{Rep}(A^\circ), \quad \forall A \in \mathrm{Alg}_2^{fd}, \tag{345}$$

and a family of invertible 2-morphisms

$$\begin{array}{ccc} (\mathrm{Rep}A)^{op} & \xrightarrow{\mathfrak{i}_A} & \mathrm{Rep}(A^\circ) \\ {\scriptstyle ((-)\otimes_A M)^{op}}\big\downarrow & \overset{\swarrow}{\mathfrak{i}_M} & \big\downarrow{\scriptstyle (-)\otimes_{A^\circ} M^\circ} \\ (\mathrm{Rep}B)^{op} & \xrightarrow[\imath_B]{} & \mathrm{Rep}(B^\circ) \end{array} \tag{346}$$

for every bimodule ${}_A M_B$, satisfying the usual pseudonaturality conditions.

Define $\mathfrak{i}_A$ to be the additive functor which

- to any right A-module V_A assigns the right A^{op}-module $V^\circ_{A^\circ} := \hom_A(V, A)_{A^{op}}$
- to any morphism $f^{op} : V_A \to W_A$ assigns $f^* : V^\circ_{A^\circ} \to W^\circ_{A^\circ}$.

Note that $\imath_A$ is an exact functor, i.e. a 1-morphism in $\mathrm{KV}_{\mathbf{k}}$.

For any bimodule ${}_A M_B$, let $\mathfrak{i}_M$ be the natural isomorphism whose component at $V \in (\mathrm{Rep}A)^{op}$ is given by the canonical isomorphism

$$(\mathfrak{i}_M)_V : V^\circ \otimes_{A^{op}} M^\circ \overset{\simeq}{\Rightarrow} (V \otimes_A M)^\circ \tag{347}$$

obtained by combining the various theorems[6] in Section 10.1. We leave to the reader to check that $\mathfrak{i}_M$ is indeed a natural isomorphism.

Lemma 10.5.1. *The family* $\mathfrak{i} := \{\mathfrak{i}_A, \mathfrak{i}_M\}_{A,M \in \mathrm{Alg}_2^{fd}}$ *gives rise to a pseudonatural transformation.*

To make $\mathfrak{i}$ into a pseudonatural adjoint equivalence, we show that $\mathfrak{i}$ is an equivalence, and invoke Theorem 10.3.8, and the subsequent remark.

We define a (pseudo) inverse $\mathfrak{i}^\square$, whose component 1-morphisms are

$$\mathfrak{i}^\square_A := \mathfrak{i}^{op}_{A^\circ}, \tag{348}$$

and whose component 2-morphisms are

$$\mathfrak{i}^\square_M := \mathfrak{i}^{op}_{M^\circ}. \tag{349}$$

To make $\mathfrak{i}$ and $\mathfrak{i}^\square$ into an equivalence pair, we consider as unit the invertible modification ϵ whose component at A is the natural isomorphism

$$\epsilon_A : 1_{(\mathrm{Rep}A)^{op}} \Rightarrow \mathfrak{i}^\square_A \circ \mathfrak{i}_A, \tag{350}$$

the component of which at $V \in (\mathrm{Rep}A)^{op}$ is the canonical isomorphism

$$(\epsilon_A)_V : V \longrightarrow (\mathfrak{i}^\square_A \circ \mathfrak{i}_A)(V) = \hom_{A^\circ}(\hom_A(V, A), A^\circ). \tag{351}$$

provided by Theorem 10.1.5 in Section 10.1.

By Theorem 10.3.8, we can consider the unique adjoint equivalence in $\mathrm{KV}_{\mathbf{k}}((\mathrm{Rep}^{co})^\circ), \mathrm{Rep} \circ (-)^\circ)$ associated to $\mathfrak{i}$, $\mathfrak{i}^\square$ and ϵ.

Definition of θ: Now we construct a modification θ whose components are invertible 2-morphisms in $\mathrm{KV}_{\mathbf{k}}$ of the following form

$$\begin{array}{ccc} \mathrm{Rep}A & \xrightarrow{(\mathfrak{i}_A)^{op}} & \mathrm{Rep}(A^\circ)^{op} \\ {\scriptstyle \mathrm{id}_{\mathrm{Rep}A}}\big\uparrow & \Big\Downarrow{\scriptstyle \theta_A} & \big\downarrow{\scriptstyle \mathfrak{i}_{A^{op}}} \\ \mathrm{Rep}A & \xrightarrow[(-)\otimes_A A]{} & \mathrm{Rep}A \end{array} \quad , \quad \forall A \in \mathrm{Alg}_2^{fd}. \tag{352}$$

[6] Recall that all the bimodules we are considering are automatically projective as left and right modules.

Namely, θ_A is a natural isomorphism between $\mathfrak{i}_{A^{op}} \circ (\mathfrak{i}_A)^{op}$ and $(-) \otimes_A A$. We choose its component at $V \in \mathrm{Rep}A$ to be the isomorphism

$$(\theta_A)_V : V^{\circ\circ} \xrightarrow{\simeq} V \otimes_A A \tag{353}$$

obtained as the following composition of canonical isomorphisms

$$V^{\circ\circ} \xrightarrow{\simeq} V \xrightarrow{\simeq} V \otimes_A A, \tag{354}$$

where the first one is the inverse of the isomorphism in (351).

We following is easily checked.

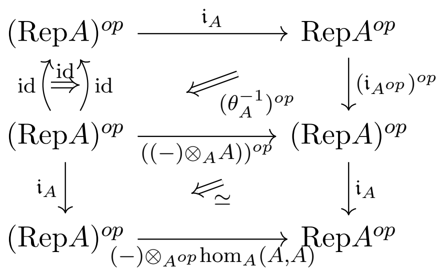
Lemma 10.5.2. *The family* $\theta := \{\theta_A\}_{A \in \mathrm{Alg}_2^{fd}}$ *defines a modification.*

We can now prove our main theorem

Theorem 10.5.3. *The bifunctor* $\mathrm{Rep} : \mathrm{Alg}_2^{fd} \to \mathrm{KV}_{\mathbf{k}}$ *equipped with the pseudonatural adjoint equivalence* $\mathfrak{i}$ *and the modification* θ *is a duality pseudofunctor.*

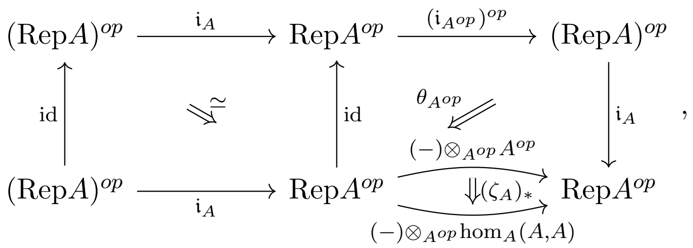
Proof. We need to check that $\mathfrak{i}$ and θ satisfy the commutativity diagram[7] in [**Shu18**]. Namely, we need to show that $\forall A \in \mathrm{Alg}_2^{fd}$ the 2-morphism

(355)



from $\mathfrak{i}_A \circ (\mathfrak{i}_{A^{op}})^{op} \circ \mathfrak{i}_A$ to $(-) \otimes_{A^{op}} \hom_A(A, A) \circ \mathfrak{i}_A$ must coincide with the 2-morphism

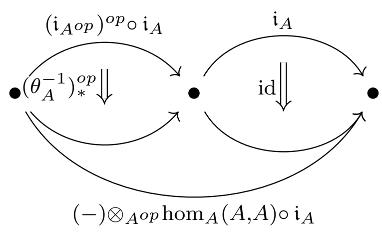
(356)



where $(\zeta_A)_*$ denotes the natural transformation induced by ζ_A.

To help the reader in the pasting procedure, one can regard the diagram (355) to be of the following globular form

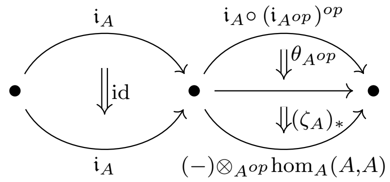
(357)



[7]Notice that the diagram in [**Shu18**] contains a small typo.

while the diagram (356) has the following form

(358)



For $V \in (\mathrm{Rep}A)^{op}$, the pasting of the diagram (355) gives rise to the following isomorphism

$$V^{\circ\circ\circ} \xrightarrow{((\theta_A^{-1})_V)^\circ} (V \otimes_A A)^\circ \xrightarrow{\simeq} V^\circ \otimes_{A^{op}} A^\circ \tag{359}$$

where the second isomorphism is provided by the inverse of (347).

On the other hand, the pasting of the diagram in (356) gives rise to the following isomorphism

$$V^{\circ\circ\circ} \xrightarrow{(\theta_{A^{op}})_{V^\circ}} V^\circ \otimes_{A^{op}} A^{op} \xrightarrow{\mathrm{id}\otimes\zeta_A} V^\circ \otimes_{A^{op}} A^\circ \tag{360}$$

To see that (359) and (360) are equal, notice that the following diagram commutes

$$\begin{array}{ccc} V^{\circ\circ\circ} & \xrightarrow{((\theta_A^{-1})_V)^\circ} & (V \otimes_A A)^\circ \\ & {\scriptstyle \psi_{V^\circ}^{-1}} \searrow & \downarrow {\scriptstyle 1_V^\circ} \\ & & V^\circ \end{array} , \tag{361}$$

where 1_V denotes the canonical isomorphism $V \to V \otimes_A A$, and ψ_V denotes the isomorphism in Section 10.1, Theorem 10.1.6. This is due to the fact that $\psi_{V^\circ} = (\psi_V^\circ)^{-1}$, and that by definition $(\theta_A)_V = 1_V \circ (\psi_V)^{-1}$. Similarly, the following diagram commutes

$$\begin{array}{ccc} (V \otimes_A A)^\circ & \xrightarrow{\simeq} & V^\circ \otimes_{A^{op}} A^\circ \\ \downarrow {\scriptstyle 1_V^\circ} & & \uparrow {\scriptstyle \mathrm{id}\otimes\zeta_A} \\ V^\circ & \xrightarrow[1_{V^\circ}]{} & V^\circ \otimes_{A^{op}} A^{op}. \end{array} \tag{362}$$

This follows from the definition of the isomorphism in (347). If we combine the two diagrams we obtain the following commutative diagram

$$\begin{array}{ccccc} V^{\circ\circ\circ} & \xrightarrow{((\theta_A^{-1})_V)^\circ} & (V \otimes_A A)^\circ & \xrightarrow{\simeq} & V^\circ \otimes_{A^{op}} A^\circ \\ & {\scriptstyle \psi_{V^\circ}^{-1}} \searrow & \downarrow {\scriptstyle 1_V^\circ} & & \uparrow {\scriptstyle \mathrm{id}\otimes\zeta_A} \\ & & V^\circ & \xrightarrow[1_{V^\circ}]{} & V^\circ \otimes_{A^{op}} A^{op} \end{array} \tag{363}$$

The upper composition corresponds to the isomorphism (359), while the lower composition corresponds to the isomorphism (360), after we notice that $1_{V^\circ} \circ \psi_{V^\circ}^{-1} = (\theta_{A^{op}})_{V^\circ}$. □

CHAPTER 11

A general setting

In this chapter we describe a general setting for the results discussed in the previous sections. We provide compact definitions of known concepts, and leave the full details of the various statements to future developments.

11.1. Algebras in finite tensor categories and their Morita bicategory

In the following $\mathcal{C}$ denotes a symmetric semisimple finite tensor[1] category. In other words, $\mathcal{C}$ is a symmetric *fusion category*; we refer to [**EGNO15**] for details concerning finite tensor categories and symmetric monoidal structures. The following definition is standard.

Definition 11.1.1. *An algebra*[2] *A in $\mathcal{C}$ is an object equipped with a multiplication $m : A \otimes A \to A$ and a unit $u : 1 \to A$ satisfying the appropriate commutative diagrams.*

Though an algebra is technically a triple (A, m, u), we refer to A as an algebra. A morphism of algebras is a morphism in $\mathcal{C}$ which is compatible with the multiplication map and the unit in an obvious manner.

Since $\mathcal{C}$ is symmetric monoidal, we can define the opposite algebra A^{op}.

Definition 11.1.2. *Let A be an algebra in $\mathcal{C}$. The* opposite algebra *A^{op} is given by equipping A with the following multiplication*

$$A \otimes A \xrightarrow{\sigma_{A,A}} A \otimes A \xrightarrow{m} A \tag{364}$$

where $\sigma_{A,A}$ denotes the braiding isomorphism of A.

We moreover have the notion of a right A-module.

Definition 11.1.3. *For an algebra A in $\mathcal{C}$, a* right A-module *is an object M in $\mathcal{C}$ equipped with a morphism*

$$M \otimes A \xrightarrow{\rho} M \tag{365}$$

called a right action of A*, which satisfies appropriate commutative diagrams.*

A left A-module is defined analogously. Similar to the rest of the chapter, when we want to emphasize that an object M in $\mathcal{C}$ is a right (resp. left) A-module, we use the notation M_A (resp. ${}_AM$).

For A and B algebras in $\mathcal{C}$, an (A, B)-bimodule M is an object in $\mathcal{C}$ which is a left A-module and a right B-module, and such that the two actions are compatible. We use ${}_AM_B$ to denote (A, B)-bimodules.

The following lemma is standard as well.

[1]We follow the convention in [**EGNO15**], and assume that the category is rigid.

[2]Another appropriate term here is "monoid". We will follow the convention of using "algebra".

LEMMA 11.1.4. *Let (M,ρ) be a right A-module. Then the morphism*

$$A \otimes M \xrightarrow{\sigma_{A,M}} M \otimes A \xrightarrow{\rho} M \tag{366}$$

equips M with the structure of a left A^{op}-module.

Similarly, any left A-module is canonically a right A^{op}-module.

A morphism between A-modules is naturally defined as a morphism in $\mathcal{C}$ which is compatible with the action ρ. In particular, right (resp. left) A-modules form a $\mathbf{k}$-linear category Mod_A (resp. ${}_A\mathrm{Mod}$). Moreover, both Mod_A and ${}_A\mathrm{Mod}$ are $\mathbf{k}$-linear abelian categories.

PROPOSITION 11.1.5. [**EGNO15**] *Let A be an algebra in a finite tensor category $\mathcal{C}$. Then Mod_A is a finite category.*

The notion of tensor product of A-modules can be expressed in general terms.

DEFINITION 11.1.6. *Let (M_A, ρ_M) and $({}_AN, \rho_N)$ be A-modules. The tensor product $M \otimes_A N$ is defined as the following coequalizer diagram*

$$M \otimes A \otimes N \underset{id_M \otimes \rho_N}{\overset{\rho_M \otimes id_N}{\rightrightarrows}} M \otimes N \longrightarrow M \otimes_A N \tag{367}$$

Since $\mathcal{C}$ is abelian, the coequalizer above is given by the cokernel of the morphism $\rho_M \otimes \mathrm{id}_N - \mathrm{id}_M \otimes \rho_N$. Hence tensor products of modules always exist.

One can show that for bimodules ${}_AM_B$ and ${}_BN_C$, the tensor product $M \otimes_B N$ carries canonically the structure of an (A,C)-bimodule, and that the usual canonical isomorphisms are guaranteed. Namely, we have that $(M \otimes_B N) \otimes_C P \simeq M \otimes_B (N \otimes_C P)$, and $A \otimes_A M \simeq M \simeq M \otimes_B B$. See [**EGNO15**] for details.

It is natural then to consider the following[3]

DEFINITION 11.1.7. *The Morita bicategory $\mathrm{Alg}_2(\mathcal{C})$ of algebras in $\mathcal{C}$ is the bicategory where:*

- *the objects are algebras in $\mathcal{C}$;*
- *the 1-morphisms are bimodules; and*
- *the 2-morphisms are morphisms between bimodules.*

Composition of 1-morphisms is given by tensoring of bimodules, and the unit 1-morphism for any algebra A is given by A itself regarded as an (A,A)-bimodule.

Notice that since $\mathcal{C}$ is symmetric monoidal, the tensor product $A \otimes B$ for algebras A and B in $\mathcal{C}$ is canonically an algebra. One can indeed show that the tensor product in $\mathcal{C}$ induces a symmetric monoidal structure on $\mathrm{Alg}_2(\mathcal{C})$. Moreover, every object A in $\mathrm{Alg}_2(\mathcal{C})$ admits a dual object with respect to this monoidal structure, namely A^{op}. More precisely, we have the following

LEMMA 11.1.8. *Let A be an algebra in $\mathcal{C}$. Then its dual is given by the opposite algebra A^{op}, and as evaluation and coevaluation we can take A regarded as an $(A \otimes A^{op}, 1_{\mathcal{C}})$-bimodule and an $(1_{\mathcal{C}}, A^{op} \otimes A)$-bimodule, respectively.*

In the lemma above, $1_{\mathcal{C}}$ denotes the tensor unit in $\mathcal{C}$. We can then consider the fully dualisable subcategory $\mathrm{Alg}_2^{fd}(\mathcal{C})$ of $\mathrm{Alg}_2(\mathcal{C})$.

[3]Beware of the different notation as in [**Lur09**]!

DEFINITION 11.1.9. *An algebra A in $\mathcal{C}$ is called* separable *if the multiplication morphism $m : A \otimes A \to A$ splits as a morphism of bimodules.*

PROPOSITION 11.1.10. *An algebra A in $\mathcal{C}$ is fully dualisable if and only if it is separable.*

PROOF. The proof is obtained by closely mimicing that in [**Sch11**]. □

REMARK 11.1.11. For $\mathcal{C} = \mathsf{Vect}_{\mathbf{k}}$, we have that $\mathrm{Alg}_2(\mathcal{C}) = \mathrm{Alg}_2$.

REMARK 11.1.12. Notice that the "finite-dimensionality" condition on A is subsumed by the fact that $\mathcal{C}$ is rigid.

The objects of $\mathrm{Alg}_2^{fd}(\mathcal{C})$ are then the separable algebras in $\mathcal{C}$, and the 1-morphisms are bimodules ${}_AM_B$ which admit right and left adjoints $({}_AM_B)^\vee$ and ${}^\vee({}_AM_B)$.

We can now consider the pseudofunctor[4]

$$(-)^\circ : \mathrm{Alg}_2^{fd}(\mathcal{C})^{co} \to \mathrm{Alg}_2^{fd}(\mathcal{C}) \tag{368}$$

defined as follows:

- to a separable algebra A it assigns A^{op};
- to a bimodule ${}_AM_B$ it assigns ${}_{A^{op}}M^\vee{}_{B^{op}}$; and
- to $f^{op} :_A M_B \to {}_AN_B$ it assigns $f^\vee :_{A^{op}} M^\vee{}_{B^{op}} \to_{A^{op}} N^\vee{}_{B^{op}}$

Following the ideas and techniques discussed in the previous sections, we formulate the following

CONJECTURE 11.1.13. *The bifunctor $(-)^\circ$ can be canonically made into a weak duality involution on $\mathrm{Alg}_2^{fd}(\mathcal{C})$.*

REMARK 11.1.14. Similar to Section 10.4, the coherence data for $(-)^\circ$ arise from the universal properties of adjoints of 1-morphisms in a bicategory.

11.2. Module categories

In this section we introduce a substitute for KV-vector spaces, in order to be able to construct a bifunctor Rep from $\mathrm{Alg}_2^{fd}(\mathcal{C})$. In the following, $\mathcal{C}$ is a category satisfying the same assumptions as in Section 11.1. Also here, our main references are [**EGNO15, DSS19**].

DEFINITION 11.2.1. *A left $\mathcal{C}$-module category is a locally finite abelian $\mathbf{k}$-linear category $\mathcal{M}$ equipped with a bilinear functor $\otimes^{\mathcal{M}} : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ together with isomorphisms witnessing the natural conditions for an action.*

A right $\mathcal{C}$-module category can be similarly defined.

DEFINITION 11.2.2. *A left $\mathcal{C}$-module functor between left $\mathcal{C}$-module categories $\mathcal{M}$ and $\mathcal{N}$ is a linear functor $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ together with isomorphisms $f_{x,m} : \mathcal{F}(x \otimes m) \simeq x \otimes \mathcal{F}(m)$ satisfying the appropriate pentagon and triangle relations.*

DEFINITION 11.2.3. *A left $\mathcal{C}$-module natural transformation between left $\mathcal{C}$-module functors $\mathcal{F}$ and $\mathcal{G}$ is a natural transformation $\eta : \mathcal{F} \to \mathcal{G}$ satisfying the condition $(id_x \otimes \eta_m) \circ f_{x,m} = g_{x,m} \circ \eta_{x \otimes m}$.*

Right $\mathcal{C}$-module functors and natural transformations can be defined similarly.

Left $\mathcal{C}$-module categories together with left exact $\mathcal{C}$-module functors and $\mathcal{C}$-module natural transformations form a 2-category $\mathrm{Mod}(\mathcal{C})$.

[4]We work under the tacit assumption that right and left adjoints for 1-morphisms have been chosen.

EXAMPLE 11.2.4. Let A be an algebra in $\mathcal{C}$. Then Mod_A is canonically a left $\mathcal{C}$-module category via the functor

$$\begin{aligned} \mathcal{C} \times \mathrm{Mod}_A &\to \mathrm{Mod}_A \\ (x, m) &\to x \otimes m \end{aligned} \tag{369}$$

REMARK 11.2.5. In [**KV94**], 2-vector spaces were introduced as module categories over $\mathrm{Vect}_{\mathbf{k}}$ with additional properties.

For $\mathcal{M}$ a left $\mathcal{C}$-module category, let $\mathrm{End}_l(\mathcal{M})$ denote the **k**-linear monoidal category of left exact $\mathcal{C}$-module functors from $\mathcal{M}$ to $\mathcal{M}$. A useful result concerning $\mathcal{C}$-module categories is the following [**EGNO15**]

THEOREM 11.2.6. *There is a bijection between structures of a left $\mathcal{C}$-module category on $\mathcal{M}$ and* **k***-linear monoidal functors $\mathcal{C} \to \mathrm{End}_l(\mathcal{M})$.*

In the following, we assume that all our module categories $\mathcal{M}$ are semi-simple as abelian categories. This is done in view of the following

PROPOSITION 11.2.7. *Let $\mathcal{M}$ be a $\mathcal{C}$-module category which is also semi-simple. Then any left exact $\mathcal{C}$-module functor $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ is exact.*

For $\mathcal{M}$ a semi-simple $\mathcal{C}$-module category, we denote with $\mathrm{End}(\mathcal{M})$ the monoidal category of exact functors.

LEMMA 11.2.8. *For $\mathcal{M}$ a semisimple $\mathcal{C}$-module category,* $\mathrm{End}(\mathcal{M})$ *is a tensor category, where duals are given by adjoints.*

Let $\mathcal{M}$ be a left (semi-simple) $\mathcal{C}$-module category, and consider the following composition of monoidal functors

$$\mathcal{C} \to \mathrm{End}(\mathcal{M}) \xrightarrow{(-)^R} \mathrm{End}(\mathcal{M})^{mp} \simeq \mathrm{End}(\mathcal{M}^{op})^{rev} \tag{370}$$

where the first functor is the one given by Theorem 11.2.6, and where $(-)^R$, $(-)^{rev}$ and $(-)^{mp}$ denote taking the right adjoint, taking the monoidally opposite category, and taking the monoidally opposite opposite category, respectively.

The monoidal functor in (370) canonically provides a monoidal functor $\mathcal{C}^{rev} \to \mathrm{End}(\mathcal{M}^{op})$, and consequently[5] a monoidal functor $\mathcal{C} \to \mathrm{End}(\mathcal{M}^{op})$. In other words, the composition above defines a left $\mathcal{C}$-module structure on $\mathcal{M}^{op}$. For notational clarity we denote by $\mathcal{M}^{\circ}$ the $\mathcal{C}$-module category $\mathcal{M}^{op}$ equipped with the module structure above. Notice that we have a canonical identification $\mathcal{M}^{\circ\circ} \simeq \mathcal{M}$ as left $\mathcal{C}$-module categories[6].

REMARK 11.2.9. Any category $\mathcal{M}$ enriched over $\mathcal{C}$ as above can be canonically given the structure of a left $\mathcal{C}$-module structure. Then $\mathcal{M}^{\circ}$ is the left $\mathcal{C}$-module category corresponding to the opposite of $\mathcal{M}$ as a $\mathcal{C}$-enriched category[7].

One can argue straightforwardly that for any (exact) $\mathcal{C}$-module functor $\mathcal{F} : \mathcal{M} \to \mathcal{N}$, the opposite functor $\mathcal{F}^{op}$ can be given the structure of a $\mathcal{C}$-module functor $\mathcal{F}^{\circ}$ between $\mathcal{M}^{\circ}$ and $\mathcal{N}^{\circ}$. The story is similar for natural transformations.

[5]Recall that $\mathcal{C}$ is symmetric monoidal, hence $\mathcal{C}^{rev} \simeq \mathcal{C}$.

[6]This is essentially due to the fact that for any pair of functors F and G between categories, $F \dashv G$ implies $G^{op} \dashv F^{op}$.

[7]Note that the opposite of an enriched category can be defined only if the enriching category is symmetric monoidal.

Let $\mathrm{Mod}^{ss}(\mathcal{C})$ denote the 2-category of semi-simple left $\mathcal{C}$-module categories, exact $\mathcal{C}$-module functors and $\mathcal{C}$-module natural transformations.

Consider the pseudofunctor

$$(-)^{\circ} : \mathrm{Mod}^{ss}(\mathcal{C})^{co} \to \mathrm{Mod}^{ss}(\mathcal{C}) \tag{371}$$

defined as follows:

- to a module category $\mathcal{M}$ it assigns $\mathcal{M}^{\circ}$;
- to a module functor $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ it assigns $\mathcal{F}^{\circ} : \mathcal{M}^{\circ} \to \mathcal{N}^{\circ}$; and
- to $\eta^{op} : \mathcal{F} \to \mathcal{G}$ it assigns $\eta^{\circ} : \mathcal{F}^{\circ} \to \mathcal{G}^{\circ}$

It is reasonable to expect then that the following is true:

CONJECTURE 11.2.10. *The bifunctor $(-)^{\circ}$ can be canonically made into a weak duality involution on $\mathrm{Mod}^{ss}(\mathcal{C})$.*

11.3. Representations

Similar to what we have done in the previous sections of this chapter, we can connect the bicategory $\mathrm{Alg}_2^{fd}(\mathcal{C})$ to $\mathrm{Mod}^{ss}(\mathcal{C})$ via the bifunctor $\mathrm{Rep}^{\mathcal{C}}$ given by taking modules over algebras. To this aim, we can use the following results [**EGNO15**]

PROPOSITION 11.3.1. *Let A be a separable algebra in a fusion category $\mathcal{C}$. Then Mod_A is a semi-simple left $\mathcal{C}$-module category.*

PROPOSITION 11.3.2. *Let A and B be algebras in $\mathcal{C}$, and let ${}_AM_B$ be an (A,B)-bimodule. Then the functor*

$$(-) \otimes_A M : \mathrm{Mod}_A \to \mathrm{Mod}_B \tag{372}$$

is a right exact $\mathcal{C}$-module functor.

We can now consider the following pseudofunctor

$$\mathrm{Rep}^{\mathcal{C}} : \mathrm{Alg}_2^{fd}(\mathcal{C}) \to \mathrm{Mod}^{ss}(\mathcal{C}) \tag{373}$$

defined as follows:

- to a separable algebra A it assigns the semi-simple $\mathcal{C}$-module category Mod_A;
- to a bimodule ${}_AM_B$ it assigns $(-) \otimes_A M : \mathrm{Mod}_A \to \mathrm{Mod}_B$; and
- to a morphism $f : M \to N$ it assigns the associated natural transformation $(-) \otimes_A M \Rightarrow (-) \otimes_A N$.

The fact that the above is a pseudofunctor is a corollary of the properties of algebra bimodules and their tensor product. Indeed, the coherence data can be defined as in Section 10.4.

We conclude with the statement of a result that we believe may be straightforwardly obtained following the lines of the special case proven in Section 10.5.

CONJECTURE 11.3.3. *The bifunctor $\mathrm{Rep}^{\mathcal{C}}$ can be canonically equipped with the data of a duality pseudofunctor between $\mathrm{Alg}_2^{fd}(\mathcal{C})$ and $\mathrm{Mod}^{ss}(\mathcal{C})$ equipped with their respective weak duality involutions.*

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